



OPTIMIZING FARES AND TRANSFER DISCOUNTS FOR A BUS-SUBWAY CORRIDOR

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Abstract. This paper aims to optimize fares and transfer discounts for public transit service along a bus-subway corridor with the consideration of effects of uncertainty in travel times and difference in stop spacing between bus and subway services on passenger behavior. The former factor is captured by the reserved time in travel cost and the latter one produces some passenger Origin–Destination (O–D) pairs along the corridor that can not be served by one mode only. This problem is formulated as a bi-level program, of which the upper level maximizes the social welfare and the lower-level capturing traveler choice behavior is a variable-demand Stochastic User Equilibrium (SUE) assignment model. A Genetic Algorithm (GA) is applied to solve the bi-level program while the Method of Successive Averages (MSA) is adopted to solve the lower-level model. A series of numerical experiments are carried out to illustrate the performance of the model and solution method. Numerical results indicate that the implementation of transfer discounts may be of great benefit to the social welfare and that the uncertainty in travel time and the difference in stop spacing play an important role in determining optimal fares and transfer discounts for the service along a bus-subway corridor.

Keywords: transfer discount, uncertainty in travel time, stop distance, bi-level program, public transit corridor.

Introduction

In the past decade, a public transit system composed of bus and subway services has been advocated as an efficient and preferred way to meet ever-increasing urban travel demand and relieve traffic congestion in metropolitan cities around the world. The comprehensive effectiveness of such a combined transit system depends not merely on a reasonable design of its topology and timetable but also on an optimal transit fare structure for it, including fares and transfer discounts. As a flexible instrument, the fare structure may influence passenger behavior directly and the operator's revenue ultimately. It is widely acknowledged that the higher fare may lead to the lower ridership. On the other hand, the low fare may result in a high subsidy from the government if the service sustains. In addition, the transfer pricing policy can also have a significant influence over passenger behavior. Therefore, optimizing fares and discounts of transfer fees between different lines and different services is a fundamental problem for the author-

ity or operator to run a public transit system. This paper considers a combined bus and subway service in parallel along a traffic corridor, which is called a bus-subway corridor hereafter, and aims to find the optimal fare structure for the corridor given the travel demand and timetable.

The optimization of transit fare structure is well investigated in the literature on transportation economics (Kocur, Hendrickson 1982; Yang, Kin 2000) and network equilibrium (Lam, Zhou 2000; Zhou *et al.* 2005; Watling 2006). The relevant research may be traced back to the 1970s, such as Nash (1978) and Glaister and Collings (1978), which proposed to treat the setting of a fare structure as an optimization problem. Originally, Nash (1978) and Glaister and Collings (1978) respectively used an elasticity based function and a linear demand function to capture the influence of travel costs on passenger behavior without regard to externalities (e.g. congestion). Spiess and Florian (1989) proposed a new assignment model for

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transit networks as an alternative way to find optimal strategies for public transit service. Li *et al.* (2009b) considered the optimal transit fare structure under different market regimes with uncertainty in a network. Tirachini *et al.* (2014) concerned multimodal pricing and optimal design of urban public transport with a focus on the interplay between traffic congestion and bus crowding. Kaddoura *et al.* (2015) found a range of values for the optimal fare and headway, due to the randomness in user behavior that is inherent to an agent-based approach. De Palma *et al.* (2015) derived the optimal time-table and the optimal prices considering crowding in public transport and its implications for pricing, seating capacity choice and optimal scheduling. There is also a great deal of effort put into consideration of capacity constraints, dwell time, fare collection systems, etc. (Tirachini, Hensher 2011; Tang *et al.* 2016). Some work in the literature focuses on the issue of coordinated pricing for different modes in a city. Li *et al.* (2009a) optimized a bus-rail transit system with feeder bus services under different market regimes. Wang *et al.* (2014) considered demand uncertainty and bounded rationality in optimizing the transit fare in an urban transportation system with public transit service and private cars. Lu *et al.* (2015) enhanced the insights into pricing mechanism for subway and parking and corresponding mode choice behavior along the corridor with elastic demand. Tang *et al.* (2017) offered an approach to integrated optimization of bus line fare and operational strategies in a public transit corridor with elastic demand and Tang *et al.* (2019) optimized limited-stop bus services with a differential fare structure, which is carried out in a public transit corridor with bus service only. However, there is a paucity of the optimization of the fare structure that includes fares and transfer discounts of the services offered in a bus-subway corridor. Moreover, these existing investigations only considered the passenger Origin–Destination (O–D) pairs that can be served or covered by either bus or subway services directly. Generally, the stop spacing is the distance between two successive stops along a transit service line. Actually, the stop spacing often differs for bus and subway services along a combined bus-subway corridor. It means that there are some added groups of passenger O–D pairs that can not be served only by one of the two services directly. These pairs can be served by a combined bus-subway service with transfer. Thus, the difference in stop spacing of the two services should be taken into account in optimizing fares and transfer discounts for a bus-subway corridor.

In addition, the travel time of either of the two transit modes is normally uncertain and variable due to congestion and other random factors. In order to analyze passenger choice behavior under stochastic travel times, Noland and Polak (2002) and Shao *et al.* (2006) built a logit model based on the random utility theory. Fujii and Kitamura (2004) and Avineri (2006) integrated the travel behavior mechanism and the cumulative prospect theory to simu-

late passenger behavior under varied uncertain factors. Yao *et al.* (2014) presented a transit network optimization method, in which the travel time reliability on the road was considered. A robust optimization model was formulated to take into account reliable transit service with stochastic travel time. Yao *et al.* (2015) formulated generalized route travel costs with the uncertainty of link travel time and the uncertainty of waiting time at a bus stop and in-vehicle congestion costs for the bus mode. Ehrgott *et al.* (2015) and Chen *et al.* (2017) considered the travel time uncertainty faced by travellers when choosing among alternative routes. Due to the distinct operational circumstances of bus and subway services, the levels of influence due to uncertainty may differ significantly between the two services. Thus, this paper also considers the influence of uncertainty in travel time of mode on passenger behavior and transforms it into an objective component of the travel cost by defining it as a reserved time in trip, just as the uncertainty effects on a transit network are measured by a function of the standard deviation of travel times on transit links in Li *et al.* (2009b).

The optimization of fare structure can be based on different objectives (Li *et al.* 2009b; Borndörfer *et al.* 2012; Tang *et al.* 2016, etc.). Whatever the forms of an objective are, the passengers' response to the fare pricing is the key part in all these investigations and is captured by means of a multinomial logit-based Stochastic User Equilibrium (SUE) model. However, the multinomial logit model has a drawback in dealing with the "common segment" of alternative paths. To avoid this, we choose the Path-Size logit model, for further details of which the reader may refer to Frejinger and Bierlaire (2007) and Ramming (2002).

The key aim of this work is to confirm advantages of optimizing the whole fare structure, especially transfer discounts, along a combined bus-subway service corridor. The other aim is to analyse impacts of consideration of new passenger O–D pairs and *uncertainty* in travel time of mode on the optimal fare structure. The two intellectual merits differentiate this work from Liu *et al.* (2017). Another intellectual merit of this work is the integration into the optimization model of *difference* in stop spacing between bus and subway services and the influence of *uncertainty* in travel time of each mode on passenger behavior. The demand along such a corridor is served by bus service only, subway service only or their combined service. Therefore, the O–D pairs along the corridor can be classified into three types according the transit service they receive.

Following this introduction, Section 1 describes the representation of a bus and subway corridor and lists basic assumptions of this investigation. Section 2 defines the components of the travel cost that impact passenger behavior. Section 3 presents model formulation and solution algorithm. Section 4 provides a set of numerical experiments to illustrate the applications of the proposed models. This paper closes with some concluding remarks.

1. Preliminaries

1.1. Assumptions

The paper considers a corridor $G(N, L)$ composed of a bus line and a subway line, as shown schematically in Figure 1, where N is the set of nodes n and L is the set of segments connecting each pair of adjacent nodes. Let W be the set of O-D pairs, w represent an O-D pair, and M be the set of paths for passengers. The bus line serves all O-D pairs along the corridor, and the subway only can serve part of the O-D pairs, because the gaps between two subway stops are usually larger than those for the bus service. In addition, there is usually a walking distance for passengers to transfer at a stop when this transfer takes place from the bus line to subway or vice versa.

Before proceeding, let us make some basic assumptions as follows:

- *transit line* – the distance between two successive stations of the subway line is twice as much as for the bus line. Each service has a constant frequency, and the average vehicle operating speed of each service is given and invariable;
- *passenger* – all passengers are assumed to be homogeneous, i.e., they have an identical value of time. There are four alternative paths for a passenger to complete her/his travel, namely, bus direct path, subway direct path and mode-combined. With regard to transfers, passengers are also assumed never to use one mode for twice in a trip, which means they can transfer once only.

1.2. Classification of O-D pairs

According to Section 1.1, transit O-D pairs in the bus and subway corridor are classified into three types in terms of their alternative paths as shown in Table 1.

Table 1. Classification of O-D pairs

O-D pairs	Transfer node	Origin	Destination	Direct path	Transfer path
		Subway line	Subway line		
Type one	yes	yes	yes	bus, subway	bus-subway, subway-bus
	no	yes	yes	bus, subway	-
Type two	no	no	no	bus	-
Type three	yes	yes	no	bus	subway-bus
		no	yes	bus	bus-subway
	no	no	no	bus	-

Type one is those O-D pairs of which both the origin and destination nodes are served by both bus and subway lines, e.g. O-D pair $(n, n+4)$ and $(n, n+2)$ in Figure 1. They can complete their trips by four alternative paths, namely, bus direct path, subway direct path, combined bus-subway transfer path or combined subway-bus transfer path if only one transfer is allowed.

Type two is those O-D pairs which are served directly by bus service, but no direct subway service whereas the combined service is not a reasonable choice, e.g. O-D pair $(n+1, n+2)$ and $(n+2, n+3)$ in Figure 1.

Type three is those O-D pairs of which both the origin and destination nodes are served by bus lines while only one of the end nodes is served by subway service, e.g. O-D pair $(n, n+3)$ and $(n+1, n+4)$ in Figure 1. In addition to the bus direct path, a passenger between an O-D pair of this type may take an alternative combined transfer path (bus-subway or subway-bus transfer path, dependent on which end node of the O-D pair is connected to the subway service).

Contrasting with the existing literature, *types two* and *three* of O-D pairs are termed added O-D pairs. It is noteworthy that only *type one* of O-D pairs exist if the stop gaps for bus and subway services are the same and both services share their stops.

2. Travel cost

Transit passengers make their path choice decision based on the travel costs of all alternatives. There are several basic components in the travel cost, e.g. walking time, waiting time, in-vehicle travel time, in-vehicle crowding discomfort, reserved time, fares and/or transfer cost, which are respectively defined as follows.

Walking time. This is the time it takes a passenger to enter and leave a subway station. This part of the journey is mainly done on foot or by walking although the lift or elevator may be used from time to time. Therefore, we term the part of journey time walking time. The walking time at a bus station T_w is disregarded here. In addition, a transit passenger also needs some time, TTr , to stations between two service lines.

Waiting time. The average waiting time TW_m a passenger spends at a station of transit mode m ($b = \text{bus}$, $s = \text{subway}$) can be calculated by:

$$TW_m = \frac{\gamma_m}{f_m}, m \in \{b, s\},$$

where: f_m is the service frequency of mode m ; the parameter γ_m depends on the distribution of transit vehicle headway and passenger arrival time. With an assumption

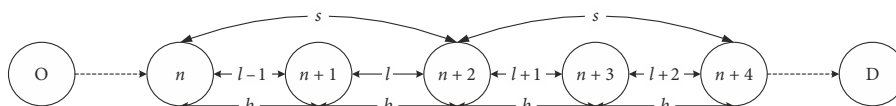


Figure 1. A bus and subway corridor

of a uniform distribution of passenger arrival and a constant transit vehicle headway, the value of γ_m is set to 0.5 (Lam, Morrall 1982).

In-vehicle travel time The average in-vehicle travel time $T_m^{L'}$ by mode m can be calculated in the following manner:

$$T_m^{L'} = \sum_{l_i \in L'} \frac{d_{l_i}}{V_m}, \quad m \in \{b, s\},$$

where: L' is the set of travel segments, $l_i \in L'$; d_{l_i} is the length of each segment; V_m is the average vehicle operating speed of mode m .

In-vehicle crowding discomfort. According to Spiess and Florian (1989), the in-vehicle crowding discomfort cost is measured in terms of generalized time units, and can be expressed in the form of Bureau of Public Roads (BPR) type function with regard to the mean vehicle travel time, passenger volume, and vehicle capacity on the line. Considering the different automobile structure of two modes, an analogical uniform type function is applied to measure the cost of in-vehicle crowding discomfort UC_m^l in mode m through segment l , which can be expressed as:

$$UC_m^l = \left(\eta_m^0 + \eta_m^1 \cdot \left(\max(0, v_m^l - f_m \cdot k_m) \right) \right) \cdot T_m^l, \quad m \in \{b, s\},$$

where: η_m^0 reflects the baseline discomfort level of mode m through segment l when the vehicle is empty; η_m^1 is the positive calibrated parameter of the in-vehicle discomfort function; v_m^l is the passenger flow using mode m on segment l ; k_m is the vehicle capacity of mode m ; T_m^l is the in-vehicle travel time by mode m on segment l .

Reserved time. Transit passengers may suffer from the delay caused by the uncertainty attributed to the operating environment of a transit mode or random factors. For the same transit mode, if the travel distance has a greater length, it is more difficult for a passenger to master the uncertainty level, which means the passenger may bear more delay. For a certain travel length, if the transit mode chosen by a passenger is easier to be interfered by uncertainty the passenger may bear more delay. Considering the uncertainty that may appear during passengers' trip, they have a rough measure of the travel time (in-vehicle travel time) prepared for trips. Normally, the difference between the rough measure and the actual measure of in-vehicle travel time is defined as a reserved time in passenger's trip. In the analysis, the reserved time $TR_m^{L'}$, is captured by a function with regard to the in-vehicle travel time $T_m^{L'}$, on the set of segments L' , by mode m , i.e.:

$$TR_m^{L'} = (\rho_m - 1) \cdot T_m^{L'}, \quad m \in \{b, s\},$$

where: $\rho_m (>1)$ is the parameter used to roughly measure the delay of mode m , from which is suffered by a passenger.

The travel cost U_M^w of direct path (s, b) for each passenger between an O–D pair w can be expressed as:

$$U_M^w = \begin{cases} u_1, & M = s; \\ u_2, & M = b, \end{cases} \quad \forall w \in W,$$

where:

$$\begin{aligned} u_1 &= \alpha_1 \cdot Tw + \alpha_2 \cdot TW_s + \alpha_3 \cdot T_s^{L^w} + \alpha_4 \cdot p_s + \alpha_5 \cdot TR_s^{L^w} + \alpha_3 \cdot \sum_{l \in L^w} UC_s^l; \\ u_2 &= \alpha_2 \cdot TW_b + \alpha_3 \cdot T_b^{L^w} + \alpha_4 \cdot p_b + \alpha_5 \cdot TR_b^{L^w} + \alpha_3 \cdot \sum_{l \in L^w} UC_b^l, \end{aligned}$$

where: the coefficients α are the reciprocal substitution factors between each cost component that is used to convert different quantities to the same unit. In this paper we set α_3 to 1.0; α_1 is the ratio of the value of walking time to the value of in-vehicle travel time, α_2 is the ratio of the value of waiting time to the value of in-vehicle travel time, α_4 is the reciprocal of the value of in-vehicle travel time, and α_5 is the ratio of the value of reserved time to the value of in-vehicle travel time. Besides, L^w is the set of segments between O–D pair w ; $T_s^{L^w}$ and $T_b^{L^w}$ are the in-vehicle travel times of the subway direct path and bus direct path, respectively; $TR_s^{L^w}$ and $TR_b^{L^w}$ are the corresponding reserved times for each direct path, respectively; p_s and p_b are the fares of subway and bus services, respectively.

It is assumed that, there are two transfer paths: bus-subway c_1 , and subway-bus c_2 , whose travel cost U_M^w , for each passenger between an O–D pair w can be expressed as follows:

$$U_M^w = \alpha_1 \cdot (Tw + TTr) + \alpha_2 \cdot (TW_s + TW_b) + TP + \begin{cases} u_1, & M = c_1; \\ u_2, & M = c_2, \end{cases} \quad \forall w \in W,$$

where:

$$\begin{aligned} u_1 &= \alpha_3 \cdot \left(\sum_{l \in L_{c1,s}^{L^w}} UC_s^l + \sum_{l \in L_{c1,b}^{L^w}} UC_b^l \right) + \alpha_3 \cdot \left(T_s^{L_{c1,s}^{L^w}} + T_b^{L_{c1,b}^{L^w}} \right) + \alpha_5 \cdot \left(TR_s^{L_{c1,s}^{L^w}} + TR_b^{L_{c1,b}^{L^w}} \right) + \alpha_4 \cdot (p_b + \lambda_s \cdot p_s); \\ u_2 &= \alpha_3 \cdot \left(\sum_{l \in L_{c2,s}^{L^w}} UC_s^l + \sum_{l \in L_{c2,b}^{L^w}} UC_b^l \right) + \alpha_3 \cdot \left(T_s^{L_{c2,s}^{L^w}} + T_b^{L_{c2,b}^{L^w}} \right) + \alpha_5 \cdot \left(TR_s^{L_{c2,s}^{L^w}} + TR_b^{L_{c2,b}^{L^w}} \right) + \alpha_4 \cdot (p_s + \lambda_b \cdot p_b), \end{aligned}$$

where: the term TP is the transfer penalty, which accounts for the resistance a passenger suffers from changing the service in addition to the walking time (Tong, Wong 1999); $L_{c1,b}^w$, $L_{c1,s}^w$ and $L_{c2,b}^w$, $L_{c2,s}^w$ are the sets of bus line segments and subway line segments in each transfer, respectively; λ_b and λ_s are the transfer discounts of the two services, respectively.

3. Model formulation

The problem refers to the situation where the whole bus and subway corridor is operated by the single operator with an objective to maximize the total social welfare of the transit system. As such, the transit operator determines the fares and transfer discounts of the two services. Furthermore, the operator's decision must be influenced by the passenger behavior, and the operator should also seek for the reasonable decision on the fare structure in order to attract more passengers. For this purpose, the model to be proposed includes two levels: the upper level aims to maximize the social welfare and the lower-level is a variable-demand SUE assignment model.

3.1. Upper sub-model

The objective of social welfare includes consumer surplus and producer profit. Although the subsidy can be provided by the authority, it is not considered in the paper.

The consumer surplus is the total benefits all consumers obtain from exchange, which can be represented by $\sum_{w \in W} \frac{g_w}{\beta}$, measured in time units following Williams (1977) and Evans (1987). The parameter $\beta (> 0)$ reflects the demand sensitivity to the expected travel cost. g_w is the total resultant passenger demand between O-D pair w , which is assumed to be elastic and is specified as a negative exponential function with respect to the expected travel cost E_w , i.e.,

$$g_w = g_w^0 \cdot \exp(-\beta \cdot E_w), \quad \forall w \in W, \quad (1)$$

where: g_w^0 is the initial potential passenger demand between O-D pair w ; according to the random utility theory, the expected travel cost E_w can be measured by the following formula (Sheffi 1985; Ben-Akiva, Lerman 1985; Oppenheim 1995):

$$E_w = -\frac{1}{\theta} \cdot \ln\left(\sum_M \exp(-\theta \cdot U_M^w)\right), \quad M \in \{b, s, c_1, c_2\}, \quad (2)$$

where: the parameter $\theta (> 0)$ describes the variation of passenger perception on travel cost in the path choice decision. In accordance with the discrete choice theory, $\theta \geq \beta$ must hold (Ben-Akiva, Lerman 1985).

The producer profit Φ is the total revenue that is generated from passenger fares minus the total operating costs, which can be expressed as:

$$\Phi = \sum_{w \in W} \left(h_b^w \cdot p_b + h_s^w \cdot p_s + h_{c_1}^w \cdot (p_b + \lambda_s \cdot p_s) + h_{c_2}^w \cdot (p_s + \lambda_b \cdot p_b) \right) - (R_s \cdot C_s + R_b \cdot C_b), \quad (3)$$

where: h_b^w , h_s^w , $h_{c_1}^w$ and $h_{c_2}^w$ are, respectively, the passenger demand of bus direct path, subway direct path and two transfer paths for O-D pair w ; C_b and C_s are the operating

costs per vehicle hour of bus and subway service, respectively; R_b and R_s are respectively the number of vehicles for bus and subway service, which can be given by:

$$\begin{aligned} R_b &= f_b \cdot CT_b; \\ R_s &= f_s \cdot CT_s, \end{aligned} \quad (4)$$

where: CT_b and CT_s are the cycle journey times for the bus and subway services, respectively.

Then the objective function of social welfare SW can be represented as:

$$\begin{aligned} SW(\mathbf{p}, \boldsymbol{\lambda}) &= \frac{\sum_{w \in W} g_w}{\alpha_4 \cdot \beta} + \\ &\sum_{w \in W} \left(h_b^w \cdot p_b + h_s^w \cdot p_s + h_{c_1}^w \cdot (p_b + \lambda_s \cdot p_s) + \right. \\ &\left. h_{c_2}^w \cdot (p_s + \lambda_b \cdot p_b) \right) - (R_s \cdot C_s + R_b \cdot C_b) \end{aligned}$$

where:

$$\begin{aligned} p_b &\geq 0, \quad p_s \geq 0; \\ 0 &\leq \lambda_b \leq 1, \quad 0 \leq \lambda_s \leq 1; \end{aligned} \quad (5)$$

$(\mathbf{p}, \boldsymbol{\lambda}) = (p_b, p_s; \lambda_b, \lambda_s)$ is the vector of decision variables.

3.2. Lower-level sub-model

The lower-level sub-model reflects the passengers' response to the given fare structure. Considering the clash between the Independence from Irrelevant Alternative (IIA) property of multinomial logit formulation and the common segments among the four alternative paths in the paper, a Path-Size Logit model (Ben-Akiva, Lerman 1985) is applied here, in which the probability P_M^w that path M is chosen between O-D pair w is defined by:

$$\begin{aligned} P_M^w &= \frac{PS_M^w \cdot \exp(-\theta \cdot U_M^w)}{\sum_M PS_M^w \cdot \exp(-\theta \cdot U_M^w)}, \\ \forall w &\in W, \quad M \in \{b, s, c_1, c_2\}, \end{aligned} \quad (6)$$

where: PS_M^w is the added term to the travel cost of alternative paths, which can be expressed by:

$$\begin{aligned} PS_M^w &= \sum_{l \in L_M} \left(\frac{t_m^l + u_m^l}{T_M^w + UC_M^w} \right) \cdot \frac{1}{\sum_M \delta_{lM}}, \\ \forall w &\in W, \quad m \in \{b, s\}, \quad M \in \{b, s, c_1, c_2\}, \end{aligned} \quad (7)$$

where: t_m^l and u_m^l are respectively the in-vehicle travel time and in-vehicle crowding discomfort on segment l of mode m ; $\delta_{lM} = 1$ if line segment l is on path M between O-D pair w , and $\delta_{lM} = 0$ otherwise; T_M^w and UC_M^w are the total in-vehicle travel time and in-vehicle crowding discomfort on path M , respectively; L_M is the set of segments of each mode on path M .

Then, the passenger flow h_M^w on path M can be computed by:

$$\begin{aligned}
 h_M^w &= g_w \cdot P_M^w, \\
 \forall w \in W, \quad M \in \{b, s, c_1, c_2\}, \quad (8)
 \end{aligned}$$

where: g_w can be obtained by Equations (1) and (2). Hence, the passenger flow on segment l of mode m , v_m^l , can be expressed by:

$$\begin{aligned}
 v_m^l &= \sum_{w \in W} \sum_M \phi_{m,M}^l \cdot h_M^w, \\
 \forall l \in L, \quad m \in \{b, s\}, \quad M \in \{b, s, c_1, c_2\}, \quad (9)
 \end{aligned}$$

where: $\phi_{m,M}^l$ is a binary variable that equals 1 if the segment l of mode m is on path M and 0 if it is not.

3.3. Solution algorithm

Considering the complexity of the bilevel optimization problem, a Genetic Algorithm (GA) with double-point crossover is adopted to solve the bi-level program while the Method of Successive Averages (MSA) (Sheffi 1985) is selected to solve the lower-level model.

The following algorithm is used to solve the previously formulated bi-level model:

- *step 1 – initialization.* Choose the values of relevant parameters in GA, including the population size, the maximum generation NG , the probabilities of performing crossover and mutation; select the range of variables p_m, λ_m ; save $n_g = 1$ for the loop time;
- *step 2 – computation.* Calculate the travel cost of each path, the path choice probability and the path passenger flow for each chromosome; count the objective function of each individual chromosome $SW(i)$; identify the maximum and minimum objective function values of population saving as SW_{\max} and SW_{\min} ; define the feature of each individual chromosome as $SW(i) - SW_{\min}$;
- *step 3 – operation.* Perform selection, reproduction, crossover and mutation procedures;
- *step 4 – verification.* Terminate the operation when the loop time reaches the maximum generation ($n = NG$), and output the data. Otherwise, let $n_g = n_g + 1$ and go to step 2.

Where the following MSA procedure is employed to complete the computation in the aforementioned step 2:

- *step 1 – initialization.* Perform a stochastic network loading procedure based on a set of initial travel cost U . Then generate a set of passenger flow on the four paths HA . Set $n = 1$;
- *step 2 – update.* Set $U^{(n)} = U(HA^{(n)})$;
- *step 3 – direction finding.* Perform a stochastic network loading procedure based on the current set of path travel costs U^n and then yield an auxiliary passenger flow pattern $HB^{(n)}$;

- *step 4 – move.* Set the new flow pattern as $HA^{(n+1)} = HA^{(n)} + \left(\frac{1}{n}\right) \cdot (HB^{(n)} - HA^{(n)})$;
- *step 5 – convergence criterion.* If the convergence is attained, stop. Otherwise, set $n = n + 1$ and go to step 2.

4. Numerical experiments and analysis

This section is to present a series of numerical experiments to illustrate the application and performance of the proposed model in handling different scenarios.

4.1. Data input

The sample corridor is shown in Figure 2. It consists of 15 nodes, one bus line that serves all nodes, one subway line that serves part of the nodes, and a bi-directional passenger demand matrix given in Table 2. The O–D pairs in dash area of Table 2 are type one O–D pairs. The distances between two successive stops of bus and subway lines are assumed to be 0.6 and 1.2 km, respectively. Each service line has its own stations. The walking time at a subway station T_w is set to 0.06 h. The walking distance to transfer is assumed to be uniform in the whole corridor and measured by transfer walking time TTr equal to 0.1 h. Other parameter values for the numerical experiments are given in Table 3. The values used for capacities, operating costs, average operating speeds and frequencies of vehicle for the two services are displayed in Table 4.

4.2. Analysis of numerical results

4.2.1. Effects of reserved time on the optimal fares

We first explore the effects of the uncertainty, which is measured roughly by passengers, reserved time in a trip, referring to type one O–D pairs in Table 2. For this purpose, it is necessary to consider three scenarios, which are described as follows:

- *first scenario:* optimize fares of the bus and subway two services when the reserved time is unconsidered;
- *second scenario:* capture passenger flow shares among different paths under the condition that fares of bus and subway services inherited from the first scenario;
- *third scenario:* optimize fares of the two services considering the reserved trip.

None of the three scenarios involve the optimization of transfer discounts, which means the transfer discounts for the two services λ_m equals 100%. The results of the first example are displayed in Table 5. Considering the first and second scenarios, the fares of bus and subway services are the same in the two scenarios. With the same travel length, the presence of reserved time in the second scenario can make a higher travel cost of all alternative paths than in the first scenario. The travel cost of bus direct paths in the second scenario is higher than that of

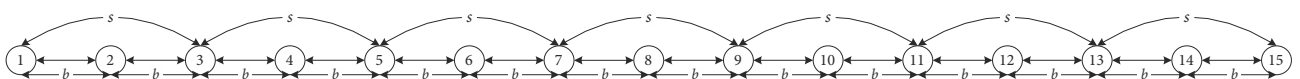


Figure 2. Bus and subway corridor

Table 2. Potential demand matrix in transit corridor [pax/h]

O \ D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	419	1786	236	1448	223	1474	198	1465	165	1497	164	1601	132	1575
2	300	0	272	480	426	350	376	454	552	312	384	250	232	240	262
3	1150	349	0	521	1799	258	1955	248	3074	182	2522	176	2159	89	3163
4	212	225	406	0	412	300	292	279	283	261	288	250	167	190	160
5	1946	275	2879	332	0	229	1390	267	1579	213	1272	206	1427	181	334
6	161	141	300	335	160	0	214	200	216	235	246	250	222	200	210
7	1456	171	2280	261	1201	193	0	215	1393	231	1546	234	1505	272	1564
8	100	170	150	132	200	210	189	0	211	160	194	200	267	220	236
9	2233	419	2210	236	1323	223	1345	198	0	204	1183	230	1451	218	1638
10	340	358	286	440	466	350	379	503	386	0	356	310	251	286	295
11	1584	218	2959	521	1774	258	1452	248	1322	221	0	271	1147	222	1414
12	250	300	312	352	409	296	296	281	262	361	271	0	260	357	351
13	1575	151	2286	317	1448	229	1484	267	1650	261	1318	295	0	362	1604
14	160	182	260	262	200	200	173	208	162	190	170	165	162	0	180
15	1645	135	1932	141	1550	205	1284	215	1140	205	1276	152	1272	201	0

Table 3. Values of parameter

Parameter	α_1	α_2	α_3	α_4	α_5	γ_b	γ_s	β	θ	ρ_b	ρ_s	η_b^0	η_b^1	η_s^0	η_s^1
Value	1.2	2.0	1.0	0.125	0.8	0.5	0.5	0.6	3.5	1.2	1.1	0.5	0.1	0.1	0.02

Table 4. Characteristics of bus and subway services

Mode	Bus	Subway
Vehicle capacity k_m [pax/veh]	120	600
Operating cost C_m [¥/veh-h]	80	900
Average operating speed V_m [km/h]	10	30
Frequency f_m [veh/h]	60	20

Table 5. Results of first experiment

Scenario	Reserved time	Fare p_m [¥]		Direct share [%]		Transfer share [%]
		Bus	Subway	Bus	Subway	
First	unconsidered	1.0	2.4	42.6	44.6	12.8
Second	considered	1.0	2.4	40.6	46.6	12.8
Third	considered	0.6	2.4	42.8	43.1	14.1

transfer paths or subway direct paths, which results in a lower passenger demand share on bus direct paths, a higher passenger demand share on subway direct paths and the same share on the two transfer paths in the second scenario comparing with the first scenario. Furthermore, the effects of on the reserved time also lead to a lower optimal fare of bus service and the same optimal fare of subway service in the third scenario comparing with the optimal fares in the first and second scenarios. Considering the second and third scenarios, the relevant time components of all alternative paths are the same in the two scenarios. In the SUE, the reduced fare components of travel cost of bus direct paths and transfer paths must be equilibrated by the increased in-vehicle crowding discomfort, which can be evidenced by the increased passenger demand shares of both bus direct paths (2.2%) and transfer paths (1.3%) in the third scenario comparing with the second scenario. Therefore, Table 5 indicates that the reserved trip time for transit service varies inversely with the optimal fare of service. In other words, the optimal fare of a transit service is lower if the service is disturbed by uncertain factors more often.

4.2.2. Effects of transfer discounts

We now look at the benefit of transfer discounts to the social welfare, and the effects of transfer discounts on passenger demand shares among all alternative. Still type one of O-D pairs in Table 2 are used for this investigation. For this purpose, it is also necessary to consider three scenarios, which can be described as follows:

- *first scenario*: optimize fares of bus and subway services with original transfer cost (i.e. transfer discount λ_m equals 100%);
- *second scenario*: optimize transfer discounts of bus and subway services with the given fares inherited from the first scenario;
- *third scenario*: optimize fares and transfer discounts of bus and subway services integrally.

The results from the second experiment are displayed in Table 6. Considering the first and second scenarios, the fares of bus and subway services are the same. With the optimization of transfer discounts in the second scenario, the relevant time and fare components of travel cost of bus

and subway direct paths are fixed, but the fare components of travel cost of bus-subway transfer paths are changed due to the optimal transfer discount. Thus, in SUE, the reduced fare components of travel cost of transfer paths must be equilibrated by the increased in-vehicle crowding discomfort in travel cost of these paths. Correspondingly, the in-vehicle crowding discomfort is reduced. The changes can be illustrated by the increased passenger demand shares of transfer paths and the reduced passenger demand shares of direct paths in the second scenario comparing with the first scenario. In addition, the optimization of transfer discounts can also benefit the passenger demand and social welfare, which increase by 84 pax/h and 115 ¥/h, respectively.

Comparing with the fares and transfer discounts in the first and second scenarios, the optimal fares in the third scenario are increased, and the resulting transfer discounts are reduced. Based on the fixed relevant time components of travel cost in all three scenarios, the increased relevant fare components of travel cost of direct paths and reduced fare components (fares and transfer pricings) of travel cost of transfer paths must make the passenger demand share of direct paths reduced and the share of transfer paths increased in the third scenario. Furthermore, the changes of passenger demand share of bus direct paths are more than those of subway direct paths due to the more increased fares of subway. For example, the optimal fare of subway service in the third scenario increases by 0.3 ¥, and the fare of bus service only increases by 0.1 ¥ comparing with that in the second scenario, which results in a larger reduced value (3.5%) in the passenger demand share of subway direct paths and a smaller reduced value (0.1%) in the passenger demand share of bus direct paths, respectively. Obviously, the social welfare in the third scenario can benefit from the integrated optimization of fares and transfer discounts in spite of the reduced passenger demand.

Table 7 displays the detailed changes in the passenger demand share of transfer paths in the given demand matrix. The transfer share in the first and third scenarios are shown as example. With the increasing travel length, the advantage of transfer paths over bus direct paths rises in terms of travel time. Therefore, the passenger demand shares of transfer paths increase with the travel length in both scenarios. With the same travel length, the passenger demand shares of transfer paths in the third scenario are higher than in the first scenario in term of the benefit of transfer discounts that are applied in the third scenario. The difference in demand share between the two scenarios can range from 5 to 8%, which shows that it is essential to adopt transfer discounts in an integrated transit corridor with aim to encourage more transfers.

4.2.3. Effects of added passenger O–D pairs considering the different stop distances of two services

We explore the interplay of passenger behavior among different O–D pair types, and further effects on the optimization of fare structure in the third experiment. For this purpose, we also look at three scenarios, which are designed as follows:

- *first scenario*: optimize fare structure without difference in stop distance, which means only type one O–D pairs (dash area in Table 2) are considered;
- *second scenario*: capture passenger demand shares among different paths considering the difference in stop distance between bus and subway services based on the fare structure inherited from the first scenario;
- *third scenario*: optimize fares and transfer discounts of bus and subway services considering the difference in stop distance between the two services.

The results from the third experiment are displayed in Table 8. Comparing with those from the first scenario, the added O–D pairs in the second scenario, namely, type

Table 6. Results of second experiment

Scenario	Fare p_m [¥]		Discount λ_m [%]		Direct share [%]		Transfer share [%]	Demand [pax/h]	SW [¥/h]
	Bus	Subway	Bus	Subway	Bus	Subway			
First	0.6	2.4	100	100	42.8	43.1	14.1	68700	1015539
Second	0.6	2.4	100	87	42.7	42.5	14.9	68784	1015654
Third	0.7	2.7	30	50	42.6	39.0	18.4	68375	1016110

Table 7. Changes in passenger demand share of transfer path [%] in second experiment: third scenario/first scenario

D \ O	1	3	5	7	9	11	13	15
1	–	0/0	21/16	23/18	26/20	30/23	33/25	35/27
3	0/0	–	0/0	21/16	23/18	26/20	30/23	33/25
5	21/16	0/0	–	0/0	21/16	23/18	26/20	30/23
7	23/18	21/16	0/0	–	0/0	21/16	23/18	26/20
9	27/20	23/18	21/16	0/0	–	0/0	21/16	23/18
11	30/23	26/20	23/17	21/16	0/0	–	0/0	21/16
13	33/25	30/23	26/20	23/18	21/16	0/0	–	0/0
15	35/27	32/25	30/23	26/20	23/18	21/16	0/0	–

two and three O-D pairs cause in-vehicle overcrowding of both services, which are illuminated by increased highest loading degree in bus and subway lines (increased by 2.9 and 4.1%, respectively). The effects of the added O-D pairs on passenger behavior of three types and passenger demand shares among several alternative paths are elaborated in the following analysis.

Passengers of type two O-D pairs must complete their trips by bus direct paths. The travel costs of bus paths increase with passengers' travel length. Thus, the travel ratio of type one O-D pairs decreases with travel length, e.g. travel ratio for O-D pairs (2, 1), (2, 4), (2, 6), (2, 8), (2, 10), (2, 12) and (2, 14) in Figure 3.

Passengers of type three O-D pairs can complete their trips by bus direct paths and transfer paths. The travel ratio of type one O-D pairs decreases with travel length. According to the assumption of transfer path in Section 1.1, the superiority of transfer paths increases with travel length comparing with bus direct paths, which result in an increasing transfer share and a decreasing bus direct share with the increasing travel length. Figure 4 shows the change in travel ratio, bus direct share and transfer share for type two O-D pairs (1, 4), (1, 6), (1, 8), (1, 10), (1, 12) and (1, 14) as example.

Types two and three O-D pairs added in the second scenario increase the passenger flow on bus line directly, which must lead to more in-vehicle crowding discomfort in the travel cost of bus direct paths. Thus, type one O-D pairs in the second scenario have a lower bus direct share comparing with the first scenario. More passengers of this type O-D pairs are likely to choose subway direct paths. Figures 5 and 6 show the different path shares for type one O-D pairs in the first and second scenarios, respectively. This shows that the subway direct path share has a significant rise, which is consistent with the results in Table 8 in which the passenger flow of subway direct path increases by 1405 pax/h. In addition, comparing with the first sce-

nario, the increased passenger flow of bus direct paths in the second scenario results from the added passenger demand of types two and three, and the increased transfer paths share in the second scenario is caused by the added transfer flow of type three and transfer flow of type one O-D pairs based on the decreased travel ratio (from 73.4 to 60.3%). Therefore, considering the difference in stop distance between bus and subway lines, the passenger behavior of added O-D pairs interacts with type one O-D pairs significantly, which imply that the difference in stop distance can not be ignored, otherwise the overcrowding may appear in transit vehicles.

Furthermore, the overcrowding in transit vehicles in the second scenario also may forebear from the higher optimal fares in the third scenario. As shown in Table 8, subject to the capacity of transit vehicles, the fares of bus and subway services rise to a level similar to transfer discounts. Comparing with the second scenario, the higher fares not only result in a lower travel ratio, but also change the passenger demand shares among different paths. For example, with the close gap of fares between the two services, the superiority of cheaper fares for bus direct paths falls down comparing with subway direct paths, which results in a decreased passenger flow in bus direct paths and increased passenger flow in subway direct paths, as shown in Table 8.

4.2.4. Sensitivity analysis of passenger demand of added O-D pairs

We carry out a sensitivity analysis of the impacts on optimal fares and transfer discounts of bus and subway services of varying the passenger demands of added O-D pairs (types two and three O-D pairs). The passenger demands of added O-D pairs in Table 2 are marked as Q^* . We optimize the fare structure with the variable Q^* , i.e. 80, 60 and 40% of Q^* . The results are displayed in Table 9.

Table 8. Results of third experiment

Scenario	Fare p_m [¥]		Discount λ_m [%]		Passenger flow [pax/h]		Transfer share [%]	Highest loading degree [%]		Travel ratio [%]
	Subway	Bus	Subway	Bus	Subway	Bus		Subway	Bus	
First	2.7	0.7	50	30	26687	29126	18.4	100.1	98.2	73.4
Second	2.7	0.7	50	30	28092	36898	18.9	104.2	101.1	60.3
Third	5.3	5.3	50	50	32718	31656	9.5	100.1	100.0	53.6

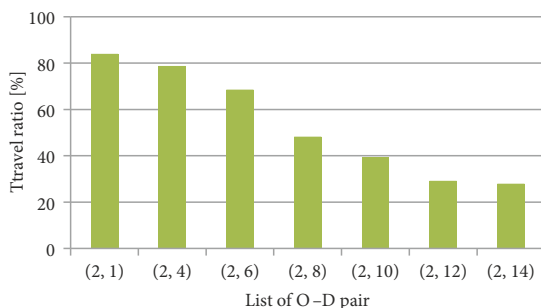


Figure 3. Travel ratio for passengers of type two O-D pairs in second scenario

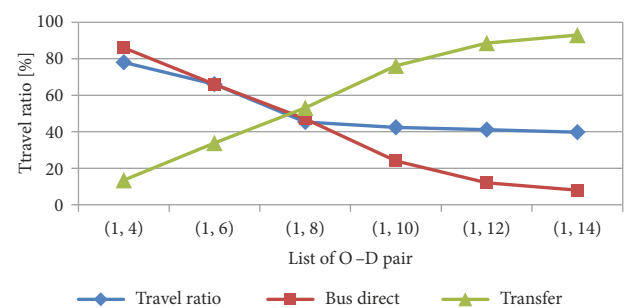


Figure 4. Travel ratio and paths shares for type three O-D pairs in second scenario

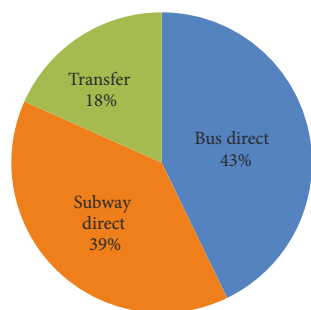


Figure 5. Paths share for type one O–D pairs in first scenario

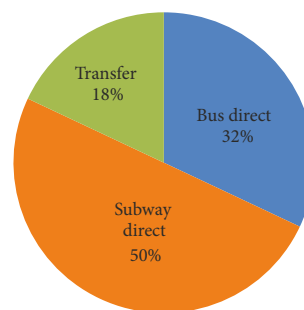


Figure 6. Paths share for type one O–D pairs in second scenario

As shown in Table 9, as the passenger demand of added O–D pairs decreases, both the fares of the two services and the transfer discount fall whereas the difference in fare between subway and bus services rises. Therefore, the passenger demand of added O–D pairs may have an indirect impact on the fare and transfer discount of the subway service and directly affect the bus service.

Table 9. Results of sensitivity analysis

Q^*	Fare p_m [¥]		Discount λ_m [%]	
	Subway	Bus	Subway	Bus
80%	4.7	4.5	60	50
60%	4.6	4.0	40	30
40%	4.0	3.0	40	20

Concluding remarks

This paper has formulated a bi-level program to optimize the fare structure (including fares and transfer discounts) along a bus-subway corridor with an objective of maximizing the social welfare. The two important factors have been considered in this optimization problem. One is the uncertainty in travel time of mode, which is due to the unstable operational environment of transit modes and/or random factors. The degree of travel time uncertainty differs from mode to mode even though they run along the same corridor. Thus, the influence of uncertainty in travel time on passenger behavior is formulated as a reserved time in the optimization model proposed in this paper, which is one of the intellectual merits of this work.

The second factor is the difference in stop spacing between bus and subway services. In a bus-subway corridor with different stop distances of the two services, passengers complete their trips by different sets of alternative paths. For example, some O–D pairs can be served by direct bus paths, some by direct subway paths and some by those paths combining subway and bus services. Thus, O–D pairs in such a corridor can be classified into three types. Type one O–D pairs are consistent with the travel groups in the existing literature, which does not focus on the difference in stop spacing between different mode services, and types two and three O–D pairs proposed in this

paper are the newly added travel groups due to the stop spacing difference issue.

Numerical experiments have been designed to explore the four issues: effects of reserved time on optimal fares, effects of transfer discounts, effects of difference in stop distance between bus and subway services and sensitivity analysis of passenger demand of the two newly added types of O–D pairs. Here is a list of new findings:

- the consideration of reserved time in a trip has a significant impact on the optimal fares of bus and subway services. If a service is more easily disturbed due to uncertain factors then the more reserved time shall be required in a trip by this service and the optimal fare of it will be lower;
- the introduction of transfer discounts of corridor transit services is of great benefit to the social welfare and transfers;
- type two O–D pairs have to choose the direct bus paths while the type three O–D pairs are likely to choose bus direct paths for short distance travel and mode-combined paths for long distance travel. The newly-added passenger O–D pairs share a certain ratio of capacity of bus service, which leads to a lower bus direct share of type one O–D pairs comparing with no consideration of the difference in stop spacing between bus and subway services. More type one passengers are likely to choose subway direct paths. In addition, these changes result in an increased bus and subway fare because of the increased congestion in bus and subway lines in the end. The observed passenger behavior indicates that it is of utmost importance to consider the difference in stop spacing between bus and subway services while optimizing public transit fare structures for different mode services along a corridor;
- the newly added types of O–D pairs have a direct impact on the optimal fare of bus service and have an indirect impact on the fare of subway service.

The major challenge in this investigation is how to assign the reasonable values of parameters accurately when we implement the case study on real-life road networks. Some stated preference surveys are needed, and it is necessary to consider the heterogeneity of passengers. Although these difficulties exist, the essential ideas of the paper are still valuable to the optimization of fare structure for a bus-subway corridor.

A possible extension of this research would be to divide passengers into groups with different trip purposes and ages because the desired travel comfort, waiting time and fare cost vary for different groups. An ongoing piece of our research is to design an applicable fare structure for groups with different purposes and/or ages along a bus-subway corridor.

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