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A NEW MODEL APPROXIMATING $M/PH/1$ QUEUING SYSTEMS DURING THE TRANSIENT PERIOD

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ABSTRACT

This paper presents an efficient approximation for $M/PH/1$ queuing systems based on the replacement of the majority of the vector valued state probabilities by a diffusion approximation. The strength of the new approximation is that it gives more accurate results than the current diffusion approximations at both high and low traffic intensities and at little extra computational cost. The accuracy of the new approximation during the transient is shown by comparing it numerically with solutions to the $M/PH/1$ system and current approaches based on the diffusion approximation.

1. INTRODUCTION

The study of queues is of general importance to the service and manufacturing industries and in the analysis of computer system performance. Many queuing models are applied to queuing systems that are operating in the steady state and the mathematical treatment of such systems is straightforward (Kleinrock [9]). It is becoming increasingly recognized that the behaviour of queuing systems during the transient is also important. Unfortunately, the transient is often simply overlooked by practitioners because of the difficulties intrinsic in the mathematics. In practice the only way to deal with transient phenomena in queues is to use approximations, either of the modelled system or of the state equations themselves. The latter using truncation and numerical techniques.

One of the techniques of probability theory that has proved attractive when it comes to implementing solutions to queuing problems on a computer has been the development of the theory of phase distributions (see [11] and [12]). These distributions are very flexible in the sense that any probability distribution can be approximated by a phase distribution to any prescribed degree of accuracy.

When queuing models are confined so that the service time and inter-arrival times are modeled using a phase type distribution then very accurate models of queuing system characteristics are obtainable. The resulting model is an infinite system of ordinary differential equations (ODEs) called the Chapman-Kolmogorov (C-K) equations. The C-K equations govern the time evolution of the probabilities that the system is in a particular state (each state represents a particular number of 'customers' in the system). The fact that the state equations are a system of ODEs makes them attractive to general practitioners.

There are two problems associated with the use of phase distributions in practice. The first is finding a suitable phase distribution to fit a given distribution ([6]-[8]) and the second is the difficulties that arise in determining solutions to the resulting C-K equations when the system is operating in a transient mode. In this paper we address issues relating to the second problem only. We note in particular that the problem of excessively large systems of equations arises, after truncation, if the phase representations of the distributions are of large order. For example to represent distributions with very small variance can require a large number of phases - the constant time service distribution is represented as an infinite phase Erlang distribution.

Diffusion approximations for queuing systems are now a well recognized approximation technique for a variety of queuing systems. In the case of $M/PH/1$ queuing systems this involves the replacement of the system of ordinary differential equations governing the probability of the state of the system, by a single partial differential equation(PDE) - the diffusion equation. The details are discussed elsewhere ([1]-[3], [5], [10] and [13]). Diffusion approximations result in models that are analytically more tractable then their $M/PH/1$ counterparts. In the case of systems requiring distributions with large number of phases, the PDE could be numerically computed (after truncation) using a finite difference scheme at much less cost than the original ODE system.

Diffusion approximations are particularly attractive when the system is congested (high traffic intensity) since the diffusion approximation is known to be very accurate for such systems. Unfortunately, when the traffic intensity is low, such approximations are inaccurate and pose serious problems.

It is important to recognize that when a diffusion approximation is employed then accuracy has been sacrificed particularly for states near the origin. It has been demonstrated, both analytically and numerically ([2], [3], [13] and [14]) that the approximation of the system of ordinary differential equations by the diffusion equation becomes increasingly more accurate in the limit when the service rate approaches the arrival rate. This is because the system spends most of its time away from the 'problematical' zero state. Several authors have suggested ways of improving accuracy for states near zero by including an instantaneous return condition (see [1] and [5]), or by moving the location of the lower boundary (see [2] and [3]).

In this paper we adopt an approach that captures the accuracy of the C-K equations for low traffic intensity and the computational efficiency offered by the diffusion approximation in high traffic intensity conditions. We propose a hybrid model - part C-K equations and part diffusion approximation. The connection between the two parts is governed by the conservation of probability. Through numerical experiments the accuracy of the model is shown to be far superior to the current diffusion approximations when used in the calculation of the state probabilities and the mean value of the number in the system.

The structure of the paper is as follows: the new model is described in Section 2; the accuracy of the model is investigated in Section 3; conclusions are presented in Section 4.

2. A NEW TRANSIENT APPROXIMATION FOR $M/PH/1$ QUEUING

One way of reducing the number of ordinary differential equations (ODEs) describing a discrete queuing processes is to replace the majority of them by a partial differential equation (PDE) representing a continuous diffusion process. Hence, when the traffic intensity is low the majority of the probability is modeled using the C-K equations modeling the first K states, the states greater than K will be replaced by a diffusion equation. The accuracy is expected to be comparable to a truncated system of C-K equations used to approximate the infinite system or replacing all the C-K equations by a diffusion approximation. This technique is expected to be accurate for both high and low traffic intensities.

The C-K equation for a $M/PH/1$ system has the following form

$$\frac{d}{dt}\pi_0(t) = \pi_0(t)(-\lambda) + \boldsymbol{\pi}_1(t)\boldsymbol{M}\boldsymbol{q}', \tag{2.1a}$$

$$\frac{d}{dt}\boldsymbol{\pi}_1(t) = \lambda\pi_0(t)\boldsymbol{p} - \boldsymbol{\pi}_1(t)(\lambda\boldsymbol{I} + \boldsymbol{B}) + \boldsymbol{\pi}_2(t)\boldsymbol{M}\boldsymbol{q}'\boldsymbol{p}, \tag{2.1b}$$

$$\frac{d}{dt}\boldsymbol{\pi}_n(t) = \boldsymbol{\pi}_{n-1}(t)\lambda\boldsymbol{I} - \boldsymbol{\pi}_n(t)(\lambda\boldsymbol{I} + \boldsymbol{B}) + \boldsymbol{\pi}_{n+1}(t)\boldsymbol{M}\boldsymbol{q}'\boldsymbol{p}, n \geq 1. \tag{2.1c}$$

The $\boldsymbol{\pi}_n(t)$ are vectors governing the state probabilities ($\pi_0(t)$ is a scalar). The bold symbols represent square matrices and vectors with dimension m, determined by the PH type distribution representing the service time [11].

The diffusion approximation to the above system is well known (see Kobayashi [10] and Newell[13])

$$f_t(x,t) = \frac{\alpha}{2} f_{xx}(x,t) - \beta f_x(x,t), \quad x > 0, \tag{2.2}$$

where $f(x,t)$ is the probability density function of x, a continuous representative of the discrete states, at time t. $\alpha = \lambda + C^2\mu, \beta = \lambda - \mu, \lambda$ is the arrival rate, μ is the mean service rate and C^2 is the squared coefficient of variation of the service time distribution.

The equation (2.2) is the subject of an appropriate boundary conditions at, or near, the origin. For a discussion of these boundary conditions see [10], [13].

In this paper we propose moving the region of the diffusion approximation (2.2) to cover the case $x \geq K+1$ where K is some integer and retain a truncated version of the C-K equations (2.1) for the states less than K.

To introduce such a model we require an ODE describing the behaviour of state K and a boundary condition governing the PDE at state $K+1$ (note that we assume that $f(x,t) \to 0$, as $x \to \infty$).

A simple proposal for a candidate equation governing state K is

$$\frac{d}{dt}\boldsymbol{\pi}_K(t) = \boldsymbol{\pi}_{K-1}(t)\lambda \boldsymbol{I} - \boldsymbol{\pi}_K(t)(\lambda \boldsymbol{I} + \boldsymbol{B}) + \mu f(K+1,t)\boldsymbol{p}. \tag{2.3}$$

The missing boundary condition for the PDE is determined by the conservation of probability,

$$\pi_0(t) + \sum_{i=1}^{K} \boldsymbol{\pi}_i(t)\boldsymbol{\varepsilon} + \lim_{\delta \to 0^+} \int_{K+1+\delta}^{\infty} f(x,t)dx = 1, \tag{2.4}$$

where $\boldsymbol{\varepsilon}$ is a m dimensional vector with all the elements 1.

To summarize, the first $K-1$ states in the model are taken directly from the C-K equations; the states greater than or equal to $K+1$ are replaced by PDE (2.2) on $x \geq K+1$, subject to auxiliary condition (2.4) and they are 'joined' by equation (2.3) at state K. For appropriate K, this model can approximate the state probabilities in both high and low traffic intensities. It is noted that to solve the resulting model we need to choose a suitable K and use numerical methods.

To apply the auxiliary condition (2.4) the standard argument is as follows. Differentiating equation (2.4) with respect to t, we obtain

$$\frac{d}{dt}\pi_0(t) + \sum_{i=1}^{K} \frac{d}{dt}\boldsymbol{\pi}_i(t)\boldsymbol{\varepsilon} + \lim_{\delta \to 0^+} \int_{K+1+\delta}^{\infty} f_t(x,t) = 0. \tag{2.5}$$

Rearranging (2.5) and noting equations (2.1) – (2.3) gives

$$-\lambda \boldsymbol{\pi}_K(t)\varepsilon + \mu f(K+1,t) - \lim_{\delta \to 0^+} \left(\frac{\alpha}{2} f_x(K+1+\delta,t) - \beta f(K+1+\delta,t) \right) = 0. \tag{2.6}$$

Unfortunately discretizing (2.6) gave poor results. In practice we obtained better results using discretization based on retaining an equation for $f_t(K+1,t)$ as in the following argument.

Discretizing the integral in (2.4) with step size equal to h, we obtain the approximation,

$$\pi_0(t) + \sum_{i=1}^{K} \boldsymbol{\pi}_i(t)\varepsilon + hf(K+1,t) + h\sum_{i=1}^{\infty} f(K+1+ih,t) = 1. \tag{2.7}$$

Differentiating the equation with respect to t, we have

$$\frac{d}{dt}\pi_0(t) + \sum_{i=1}^{K} \frac{d}{dt}\boldsymbol{\pi}_i(t)\varepsilon + hf_t(K+1,t) + h\sum_{i=1}^{\infty} f_t(K+1+ih,t) = 0. \tag{2.8}$$

Rearranging (2.8) and noting equations (2.1), (2.3) and discretizing (2.2) on x with a standard explicit second order finite-difference results in

$$\begin{aligned} f_t(K+1,t) =& \frac{1}{h}(\lambda \boldsymbol{\pi}_K(t)\varepsilon - (\mu + \frac{\alpha}{2h} + \frac{\beta}{2})f(K+1,t) \\ &+ (\frac{\alpha}{2h} - \frac{\beta}{2})f(K+1+h,t)). \end{aligned} \tag{2.9}$$

A simple numerical scheme for the model is shown in the following. Let τ be a time step size, h be the step size for variable x. We approximate $\boldsymbol{\pi}_j(t) \simeq \boldsymbol{\pi}_j^n$ and $f(x,t) \simeq f_j^n$ where $t = n\tau, x = K+1+mh, m$ or $n = 0,1,2,....$ We approximate the new model using Euler time stepping for the ODE and the standard explicit second order Finite-difference (FD) scheme for the PDE.

$$\pi_0^{n+1} = \pi_0^n + \tau(-\lambda\pi_0^n + \boldsymbol{\pi}_1^n \boldsymbol{M}\boldsymbol{q}'), \tag{2.10a}$$

$$\boldsymbol{\pi}_1^{n+1} = \boldsymbol{\pi}_1^n + \tau(\lambda\pi_0^n \boldsymbol{p}\boldsymbol{I} - \boldsymbol{\pi}_1^n(\lambda\boldsymbol{I} + \boldsymbol{B}) + \boldsymbol{\pi}_2^n \boldsymbol{M}\boldsymbol{q}'\boldsymbol{p}), \tag{2.10b}$$

$$\boldsymbol{\pi}_m^{n+1} = \boldsymbol{\pi}_m^n + \tau(\boldsymbol{\pi}_{m-1}^n \lambda\boldsymbol{I} - \boldsymbol{\pi}_m^n(\lambda\boldsymbol{I} + \boldsymbol{B}) + \boldsymbol{\pi}_{m+1}^n \boldsymbol{M}\boldsymbol{q}'\boldsymbol{p}),\ m \le K-1, \tag{2.10c}$$

$$\boldsymbol{\pi}_K^{n+1} = \boldsymbol{\pi}_K^n + \tau(\boldsymbol{\pi}_{K-1}^n \lambda\boldsymbol{I} - \boldsymbol{\pi}_K^n(\lambda\boldsymbol{I} + \boldsymbol{B}) + \mu f_{K+1}^n), \tag{2.10d}$$

$$f_{K+1}^{n+1} = f_{K+1}^n + \frac{\tau}{h}(\lambda\boldsymbol{\pi}_K^n\varepsilon - (\mu + \frac{\alpha}{2h} + \frac{\beta}{2})f_{K+1}^n + (\frac{\alpha}{2h} - \frac{\beta}{2})f_{K+2}^n), \tag{2.10e}$$

$$\begin{aligned} f_m^{n+1} =& f_m^n - \frac{\tau\beta}{2h}(f_{m+1}^n - f_{m-1}^n) \\ &+ \frac{\tau\alpha}{2h^2}(f_{m+1}^n - 2f_m^n + f_{m-1}^n),\ m > K+1. \end{aligned} \tag{2.10f}$$

It is clear that for the $M/M/1$ queue there is probably little to be gained, though the possibility of very large gaps between the mesh points for large x (representing a computational saving) is possible. On the other hand, the discretized PDE can offer a substantial saving over the ODE system for many $M/PH/1$ queuing systems - especially if the phase distributions are of large order.

In order to apply the model we are required to obtain discrete probabilities from the continuous density function $f(x,t)$. A variety of techniques were tried following the ideas in ([1], [3], [5] and [10]), and for the above scheme best results were obtained by using

$$\pi_{K+n}(t) = \int_{K+n}^{K+n+1} f(x,t)dx, \quad n = 1, 2, ... \tag{2.11}$$

$$L(t) = \sum_{n=0}^{K} n\boldsymbol{\pi_n}\boldsymbol{\varepsilon} + \int_{K+1}^{\infty} xf(x,t)dx. \tag{2.12}$$

3. ACCURACY OF THE APPROXIMATION

In this section the accuracy of the approximation (2.10) is numerically investigated by comparing its performance with current approaches using diffusion approximations ([1], [3], [5] and [10]).

The conclusions of this section are based on 5 Erlangian and 4 Hyper-exponential service time distributions with $C^2 = 0.5, 0.3333, 0.2, 0.1, 0.0125$ and $C^2 = 2, 3, 5, 10$ respectively, and have been further validated on other more general $M/PH/1$ systems. Only a limited number of examples of these experiments are presented here - full detailed results are provided in Gao[4].

To compare the approximations, the following performance measures are used.

$\pi_i(t)$: The scalar probability of i customers in the system at time t.

$L(t)$: The mean number of customers in the system at time t.

In the steady state analysis we drop the parameter t to indicate the corresponding steady state values. To measure the error in these performance parameters as compared with the exact results (the exact results for the $M/PH/1$ system are calculated using the Pollachek-Khinchine formula [9] both the error and relative percentage error are used. They are defined as

$$\text{error} = \text{approximate result} - \text{exact result},$$

$$\text{relative percentage error} = \frac{\text{approximate result} - \text{exact result}}{\text{exact result}} \times 100\%.$$

In the steady state the errors in π_i and L are denoted by $\triangle\pi_i$ and $\triangle L$, and relative percentage errors in L, denoted by $e\%$.

In the analysis of transient performance the maximum absolute error (Δ_m) and the maximum relative percentage errors (Δ_n) of $L(t)$ for all time are used. These are defined, respectively, as

$$\Delta_m = \max_{t\in[0,\infty)} \{|\Delta L(t)|\}, \quad \Delta_n = \max_{t\in[0,\infty)} \{|e(t)|\}.$$

Steady state performance. We present the steady state error comparisons for the approximations generated by the new model with Duda [1], Filipiak [3], Gelenbe [5] and Kobayashi [10] for the $M/E_3/1$ system in Table 1. Table 1 is for the case when $\rho = 0.5$ and 0.95 and the new model with various K.

It is noted from the Table 1 that the maximum errors of the probabilities in the previous diffusion approximations always occurs in π_0. A similar pattern is observed from the new model in which the maximum errors for the probabilities now occur in π_{K+1}. Filipiak's [3] approximation for π_0 is worst for all cases but his L is more accurate for those approximations where the traffic intensity is close to 1.

From the results of extensive numerical experiments (of which Table 1 is an example) we have the following conclusions.

Table 1.
Errors comparison of π_i and L at steady state in $M/E3/1$ systems

	K	$\Delta\pi_0$	$\Delta\pi_1$	$\Delta\pi_2$	$\Delta\pi_3$	$\Delta\pi_4$	$\Delta\pi_5$	ΔL	$\%e$
$\rho = 0.50$									
K		0.000	0.055	-0.021	-0.018	-0.009	-0.004	-0.118	-14.139
G		0.000	0.055	-0.021	-0.018	-0.009	-0.004	0.083	10.000
F		-0.170	0.174	0.015	-0.007	-0.006	-0.003	0.126	15.108
D		0.000	0.055	-0.021	-0.018	-0.009	-0.004	-0.167	-20.000
N	2	-0.002	-0.001	-0.001	0.014	-0.003	-0.003	-0.009	-1.087
	3	-0.002	-0.001	0.000	0.000	0.006	-0.001	0.003	0.328
	4	-0.001	0.000	0.000	0.000	0.000	0.002	0.003	0.342
	5	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.197
$\rho = 0.95$									
K		0.000	0.007	0.002	0.001	0.000	0.000	-0.311	-2.392
G		0.000	0.007	0.002	0.001	0.000	0.000	0.158	1.220
F		-0.024	0.009	0.004	0.002	0.002	0.002	0.015	0.111
D		0.000	0.007	0.002	0.001	0.000	0.000	-0.317	-2.439
N	2	0.001	0.001	0.001	0.001	0.001	0.001	-0.217	-1.673
	3	0.000	0.000	0.000	0.000	0.001	0.001	-0.162	-1.248
	4	0.000	0.000	0.000	0.000	0.000	0.001	-0.135	-1.040
	5	0.000	0.000	0.000	0.000	0.000	0.000	-0.117	-0.904
	6	0.000	0.000	0.000	0.000	0.000	0.000	-0.103	-0.794
	7	0.000	0.000	0.000	0.000	0.000	0.000	-0.091	-0.698
	8	0.000	0.000	0.000	0.000	0.000	0.000	-0.080	-0.612
	9	0.000	0.000	0.000	0.000	0.000	0.000	-0.070	-0.536
	10	0.000	0.000	0.000	0.000	0.000	0.000	-0.061	-0.467

K:=Kobayashi, G:=Gelenbe, F:=Filipiak, D:=Duda, N:= New Model

1) Generally speaking, as K gets larger the numerical results become more accurate. It is also true that as ρ increases the errors rapidly decrease for small K.

2) Any specified accuracy of the results can be achieved by choosing an appropriate K.

3) Generally speaking, when $K = 2(C^2 < 1)$ and $K = 6(C^2 > 1)$ the approximation in the new model can reach reasonable accuracy, the relative percentage errors for L are smaller than 4.3%, which is accurate enough for most practical applications.

Transient performance. Numerical experiments were done solving the model (2.10). A time step increment of τ=0.1 was used and the model investigated using various space step sizes h. Surprisingly, numerical comparisons show that, when the space discretization is 1, the results are closest to the exact solution of the C-K equations. By giving the errors in $\pi_i(t)$ and $L(t)$, and the maximum absolute errors in $L(t)$ for all time, we present a comparison of the approximations generated by the new model with Duda [1] and Filipiak [3].

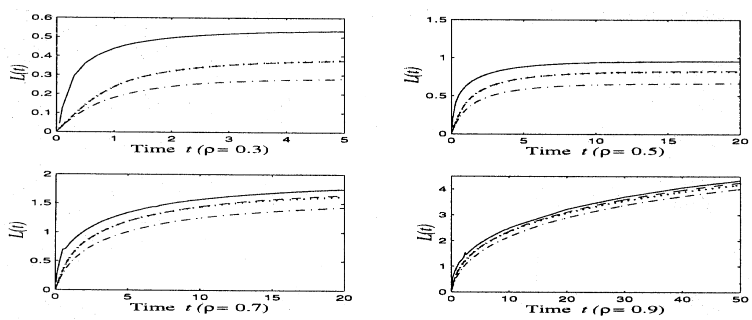

Figure 1. Comparison of the approximations for $L(t)$ in $M/E_3/1$ system from the new model($K = 2$). - - Exact, — Filipiak, —· Duda, ··· new model($K = 2$)(Note nearly indistinguishable from exact solution)

The comparison of the approximations to $L(t)$ in $M/E_3/1$ systems with $\rho = 0.3, 0.5, 0.7$ and 0.9 is shown in the following Figure 1, from which we see that the new model with $K = 2$ is more accurate than the others for all the traffic intensities over the time period shown.

Figure 2 ($\rho = 0.7$) presents the error comparisons to the transient approximations generated by the new model ($K = 2$) with Duda's [1] and Filipiak's [3] for the $M/E_3/1$ system. The system is considered starting from empty. Table 2 presents maximum absolute errors and maximum relative percentage errors of the approximation to $L(t)$, for all the time, in new model($K = 2$), Duda [1] and Filipiak [3] with $\rho = 0.5, 0.95$ and various K. From Table 2 and Figure 2, we have the following conclusions.

1*) When $K = 2$ the maximum absolute errors and maximum absolute relative errors of the approximation to $L(t)$ in the new model are much smaller than those of Duda [1] and Filipiak [3].

2*) When K gets larger, the approximation in the new model becomes more accurate over all periods of time and for all traffic intensities.

3*) Generally speaking, when $K = 2$ the approximation in the new model is more accurate than those in [1] and [3] over all periods of time and for all

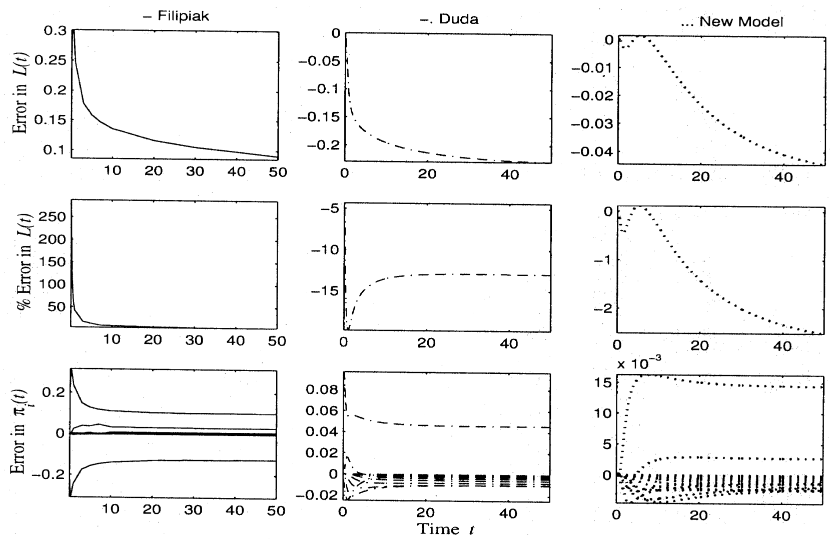

Figure 2. Errors comparison of the approximations for $L(t)$ and $\pi_i(t)$ for the $M/E_3/1$ system with $\rho = 0.7$. Note the magnitudes on the vertical axes clearly show the superiority of the new model.

traffic intensities, except some measures in [3] in some special cases.

Extensive numerical experiments [4] verify these conclusions for apparently all C^2.

4. CONCLUSION

In this paper a new approximation scheme has been presented for modelling the $M/PH/1$ queuing system. The scheme is straightforward to implement and is computationally much more efficient than solving a truncated version of the C-K equations for a given level of accuracy. The advantages of the new scheme in heavy traffic and for modelling PH service time distribution with a large number of phases has been identified.

The new model gave steady state results which were within 4.3% of the correct value for modest values of K and this can be improved upon if required (simply increase K). The errors arising in the approximation during the transient period were smaller than the other diffusion approximations discussed in the literature. The success of the new scheme has been verified by a large range of numerical experiments of which only a small sample have been presented in this paper – the details to appear in the PhD thesis of Gao [4].

The use of hybrid schemes of the type described in the paper has easy extension to any systems in which a diffusion approximation can be applied to a system with phase type distributions. For example multiple server systems should not pose any difficulties. The real value of the new model is expected

Table 2.
Comparison of the maximum absolute error in $L(t)$ $(M/E3/1$ system)

	ρ	K	Δ_m	Δ_n		ρ	K	Δ_m	Δ_n
F	0.50		0.265	303.341	F	0.95		235.747	248384.170
D	0.50		0.167	21.939	D	0.95		0.242	17.887
N	0.50	2	0.003	1.087	N	0.95	2	0.217	1.673
	0.50	3	0.006	0.695		0.95	3	0.162	1.248
	0.50	4	0.004	0.433		0.95	4	0.135	1.040
	0.50	5	0.002	0.220		0.95	5	0.117	0.904
	0.50	6	0.001	0.103		0.95	6	0.103	0.794
	0.50					0.95	7	0.091	0.698
	0.50					0.95	8	0.080	0.612
	0.50					0.95	9	0.070	0.536
	0.50					0.95	10	0.061	0.467
	0.50					0.95	11	0.053	0.406
	0.50					0.95	12	0.046	0.351

F:=Filipiak, D:=Duda, N:= New Model

to be in the analysis of problems in which the queuing system is subject to a Poisson arrival process with time varying mean. Initial experiments in this area are proving very encouraging.

REFERENCES

[1] A. Duda. Diffusion Approximations for Time Dependent queuing Systems. *IEEE Journal on Selected Areas in Communications*, **SAC-14** (6), 1986, 905 – 918.

[2] J. Filipiak. Diffusion-Equation Model of Slightly Loaded M/M/1 queue. *Operations Research Letters*, **2** (3), 1983, 134 – 139.

[3] J. Filipiak. Location of a Boundary for a Diffusion Equation Model of a Single Server queue. *Performance Evaluation*, **6** 1986, 205 – 218.

[4] H. Gao. Approximations of Simple Queueing System Transients. *PhD Thesis, Glasgow Caledonian University* (to appear) .

[5] E. Gelenbe. On approximate computer system models. *Journal of the Association for Computing Machinery*, **22** (2), 1975, 261 – 269.

[6] M. A. Johnson and M. R. Taafe. Matching Moments to Phase Distributions : Mixtures of Erlang Distributions of Common Order. *Commun. Statist.-Stochastic Models*, **5** (4), 1989, 711 – 743.

[7] M. A. Johnson and M. R. Taaffe. Matching Moments to Phase Distributions: Non Linear Programming Approaches. *Commun. Statist - Stochastic Models*, **6** (2), 1990, 259 – 281.

[8] M. A. Johnson and M. R. Taaffe. An Investigation of Phase-Distributions Moment-Matching Algorithms for Use in Queueing Models. *Queueing Systems*, **8** 1991, 129 – 148.

[9] L. Kleinrock. *Queueing System I: Theory.* Wiley, New York, 1975.

[10] H. Kobayashi. Application of the Diffusion Approximation to Queueing Networks I-Equilibrium Queue Distributions. *Journal of the Association for Computing Machinery*, **21** (2), 1974, 316 – 328.

[11] L. Lipsky. *Queueing Theory: A Linear Algebraic Approach.* Macmillan Publishing Company, New York, 1992.

[12] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach.* Dover Publications, INC., New York, 1994.

[13] G. F. Newell. *Applications of Queueing Theory.* Chapman & Hall, London, 1971.

[14] M. Reiser and H. Kobayashi. Accuracy of the Diffusion Approximation for some Queueing Systems. *IBM J. Res. Develop*, **74** 1974, 110 – 124.

Masinio aptarnavimo sistemos $M/PH/1$ aproksimavimo perėjimo periodu naujas modelis

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Straipsnyje pateikta efektyvi masinio aptarnavimo sistemos $M/PH/1$ aproksimacija, kuri gaunama naudojant daugumos būsenų tikimybių difuzinę aproksimaciją. Šios aproksimacijos privalumas tas, kad gaunami tikslesni rezultatai, negu naudojant įprastines difuzines aproksimacijas, tiek esant mažiems, tiek dideliems paraiškų intensyvumui. Be to, pakanka mažiau skaičiavimų. Šios naujos aproksimacijos tikslumas perėjimo būsenoje parodomas lyginant sistemos $M/PH/1$ sprendinius su sprendiniais, kurie gaunami naudojant įprastinę difuzinę aproksimaciją.