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MULTISTEP DEGENERATE MATRIX METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

Two of the simplest general schemes of the degenerate matrix method in the multistep mode are considered. The stability function for these methods is computed by the residue theory in the complex plane. Performances of uniformly and nonuniformly distributed nodes in the standardized interval are compared.

1. INTRODUCTION

The degenerate matrix method (DM method) is a special computing scheme for numerical solutions of initial value problems of ODE. It is based on the use of matrices of two types: matrices for derivatives $\mathbf{\Delta}_N$ and their quasi-inverses $\mathbf{B}_N$ at the fixed system of N nodes on the standardized interval $[-1, 1]$. Matrices for derivatives are always degenerate, but their quasi-inverses matrices which are not unique for given $\mathbf{\Delta}_N$ can be both degenerate and nondegenerate. Applications of DM-methods in the mode of Runge-Kutta schemes give usually at once a whole set of methods which depend on the step H, on the number N of nodes, on a choice of the nodes themselves, and also on N arbitrary constants. The special choice of constants and nodes on the standardized interval $[-1, 1]$ gives a phenomena that the norm of the quasi-inverse matrix $\mathbf{B}_N$ is the minimal possible, equal to 2, and does not dependent on the number N. Therefore these schemes which depend on two parameters H and

N have a property that the choice of the step ensures a convergence of the iterations procedure for discretized equations, but the choice of number N - only the precision. Therefore, DM-method with such schemes makes always theoretically possibly to obtain numerical solutions with an arbitrary precision even for stiff equations. Such possibilities along with the general schemes of DM-method were demonstrated in [1]. Unfortunately, stiff equations possessing very large Lipschiz constants require a long time for calculations in this mode because of the necessity in this case to choose a very small step H ensuring the convergence of iterations for discretized equations. Nevertheless, these schemes can be always applied to the finding of accurate starting values for multistep methods. In practice, it is necessary to apply such methods which have an admissible lesser precision but require essentially a shorter time for calculations. One of that possibilities is using of DM-methods in the multistep mode. In this article we will analyze only two methods.

1. Applying for calculations only the last row of the matrix for the derivatives $\mathbf{\Delta}_N$ with arbitrary distributed nodes on the standardized interval $[-1, 1]$.

2. Applying for calculations only the last row of quasi-inverse matrix $\mathbf{B}_N$ with arbitrary distributed nodes on the standardized interval $[-1, 1]$. Besides of depending on nodes each method depends also on one arbitrary constant which influences on the stability and precision of them.

A special attention is devoted to the discovering and investigation of stability functions of these methods. To this end methods which are similar to ones in [2] are used mainly.

Aside from these two methods there exist also another computing schemes in the multistep mode. For example, using two last rows of pseudoinverse matrices it is possible to obtain and to analyze different Adam's methods simply.

2. SCHEMES DIRECTLY USING ONLY MATRICES FOR DERIVATIVES

Let seek step by step the numerical solution for the initial value problem of ODE

$$y' = f(t, y),\ \ y|_{t=a} = \alpha_0,\ t \in [a, b], \tag{2.1}$$

where $y, \alpha_0, f \in +\mathbf{R}^m$. Let's choose a length H of the mold and $t_0 < t_1 < \ldots < t_N < t_{N+1} = t_0 + H$ as nodes in the interval $[t_0, t_0 + H]$. At the first step we have $t_0 = a$. By means of the linear substitution $x = (2t - 2t_0 - H)/H$ we reduce the problem (2.1) in $[t_0, t_0 + H]$ to

$$y'_x = \frac{H}{2} F(x, y),\ \ y|_{x=-1} = \alpha \tag{2.2}$$

in the interval $[-1,1]$, where $F(x,y) = f(t_0 + 0.5H(x+1), y)$; $\alpha = y(t_0)$. For the problem (2.2) the length of the mold is equal to 2 and the nodes are

$$-1 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1. \tag{2.3}$$

The problem (2.2) is called the problem on the standardized interval $[-1,1]$. Let $\mathbf{\Delta}_{N+2}$ be a matrix for derivatives constituted with respect to nodes (2.3) and let we know the solution of (2.2) at nodes $x_0 < x_1 < \ldots < x_N$ as the starting values for the multistep method. Then

$$y'(x_{N+1}) \equiv y'(1) = \sum_{k=0}^{N+1} \delta_k y(x_k), \tag{2.4}$$

where $\delta_k := \delta_{N+1,k}$, $k = 0,1,\ldots,N+1$, are elements of the last row of the matrix $\mathbf{\Delta}_{N+2}$. Therefore, using formulas from [1]

$$\delta_{N+1} = \frac{p''_{N+2}(1)}{2p'_{N+2}(1)}, \quad p_{N+2}(x) = (x^2-1)q_N(x), \tag{2.5}$$

$$q_N(x) = \prod_{k=1}^{N}(x - x_k), \quad \delta_k = \frac{p'_{N+2}(1)}{(1-x_k)p'_{N+2}(x_k)}, \; k = 0,1,\ldots,N, \tag{2.6}$$

we obtain the following implicit equation for the vector $y(x_{N+1}) \equiv y_{N+1}$:

$$-\delta_{N+1}y_{N+1} + \frac{H}{2}F(x_{N+1}, y_{N+1}) = \sum_{k=0}^{N} \delta_k y_k, \quad y_k \equiv y(x_k), \tag{2.7}$$

if $y_k, k = 0,1,\ldots,N$, as starting values are known. The equation (2.7) can be solved usually by the iterative loop. On the next step we suppose that $x \to x+h$, $x_k \to x_k + h$, where $h = x_{N+1} - x_N$, and again use (2.6) recalculating $y(x_k+h)$ by interpolation formulas. The coefficients δ_k, $k = 0,1,\ldots,N+1$, and H do not change. Then the calculations are repeated.

Remark 2.1. 1. We choose the standardized interval $[-1,1]$ since classical polynomials whose zeroes we use usually as nodes have such domain by definition. In [2] the standardized interval $[0,1]$ instead of $[-1,1]$ is used, too.

2. Coefficients of matrix $+\mathbf{\Delta}_{N+2}$ are preserved if we do not change the length of the mold H. Otherwise we replace H with H_{new}/H_{old} in (2.2).

3. This computing scheme (2.7) is analogously with methods known in literature, for example in [2], as the back differentiation formulas (BDF) with nonuniformly distributed nodes in the general case.

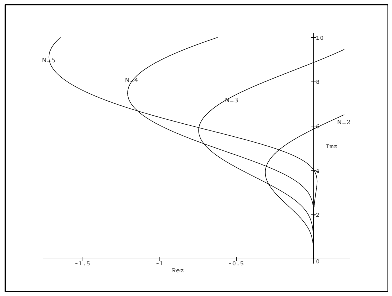

Figure 1. The borders of stability regions in the case of uniformly distributed nodes

Now we will investigate the stability of the method given by (2.7). To this end we consider the Dalquist test equation in the standardized interval $[-1,1]$

$$y' = \frac{\lambda H}{2} y, \quad y|_{x=-1} = y_0, \tag{2.8}$$

having the precise solution

$$y(x) = y_0 \exp[z(x+1)], \quad z = \frac{\lambda H}{2} \in \mathbf{C}. \tag{2.9}$$

An application of the scheme (2.7) for (2.8) gives the equation

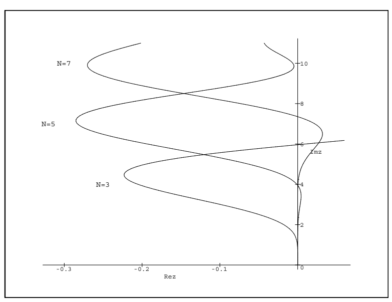

Figure 2. The borders of the stability regions for Chebyshev nodes

$$(z - \delta_{N+1}) y_{N+1} = \sum_{k=0}^{N} \delta_k y_k. \tag{2.10}$$

Using (2.5), (2.6) and the assumption that $y(x)$ is an analytic function we can rewrite (2.10) in the form of residues since all nodes $x_k,\ k = 0, 1, \ldots, N$, are simple zeroes of the polynomial $p_{N+2}(x)$.

$$(z - \delta_{N+1}) \frac{y_{N+1}}{y_0} = p'_{N+2}(1) \sum_{k=0}^{N} Res \left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(1-\xi)}, x_k \right], \tag{2.11}$$

where $\bar{y}(\xi) = y(\xi)/y_0$. Taking into account that the sum of residues on the compactified complex plane is equal to zero, we obtain instead of (2.11)

$$\frac{y_{N+1}}{y_0} = \frac{p'_{N+2}(1)}{z-\delta_{N+1}} \left\{ Res\left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(\xi-1)}, 1\right] + Res\left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(\xi-1)}, \infty\right]\right\}. \tag{2.12}$$

We define the stability function $R(z)$ by equality $R(z) = y_{N+1}/y_0$ or

$$R(z) = \frac{p'_{N+2}(1)}{z-\delta_{N+1}} \left\{ Res\left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(\xi-1)}, 1\right] + Res\left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(\xi-1)}, \infty\right]\right\}. \tag{2.13}$$

The computing of residues in (2.13) with $\bar{y}(\xi) = \exp[z(1+\xi)]$ leads to

$$Res\left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(\xi-1)}, 1\right] = \frac{(z-\delta_{N+1})\exp(2z)}{p'_{N+2}(1)}, \tag{2.14}$$

$$Res\left[\frac{\bar{y}(\xi)}{p_{N+2}(\xi)(\xi-1)}, \infty\right] = -\frac{\exp(z)z^{N+2}}{(N+2)!}\sum_{k=0}^{\infty}\frac{c_k z^k}{(N+2)_k}, \tag{2.15}$$

where $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$, $(\alpha)_0 := 1$ and c_k are coefficients of the following Laurent series

$$\frac{1}{p_{N+2}(\xi)(\xi-1)} = \xi^{-N-3}\sum_{k=0}^{\infty}\frac{c_k}{\xi^k}, \quad |\xi| > 1 \tag{2.16}$$

at $\xi = \infty$. In particular, we always have that $c_0 = c_1 = 1$ always. Substituting (2.14), (2.15) and (2.16) into (2.13) gives

$$R(z) = \exp(2z) - \frac{p'_{N+2}(1)z^{N+2}\exp z}{(z-\delta_{N+1})(N+2)!}\left[1 + \frac{z}{N+3} + \sum_{k=2}^{+\infty}\frac{c_k z^k}{(N+3)_k}\right]. \tag{2.17}$$

Since $\Psi(z) = \exp(2z) - R(z)$ is an error function we have the representation

$$\Psi(z) = \frac{p'_{N+2}(1)z^{N+2}\exp z}{(z-\delta_{N+1})(N+2)!}\left[1 + \frac{z}{N+3} + \sum_{k=2}^{+\infty}\frac{c_k z^k}{(N+3)_k}\right]. \tag{2.18}$$

By (2.18) we conclude that an order of the method is always $N+2$, but the error constant is

$$\beta_N = -\frac{p'_{N+2}(1)}{\delta_{N+1}(N+2)!}.$$

Example 2.1. There are some samples of the borders of the stability regions for the function $R(z)$ obtained by setting $|R(z)| = 1$.

1°. In the Fig. 1 borders of the stability regions for uniformly distributed $N = 2, 3, 4, 5$ nodes in the standardized interval $[-1, 1]$ are shown.

2°. In the Fig. 2 these borders for $N = 3, 5, 7$ nodes as zeroes of Chebyshev polynomials of the second kind are shown.

3°. In the Fig. 3 the borders of the stability regions for uniformly distributed nodes and Chebyshev nodes are compared in the case $N = 5$.

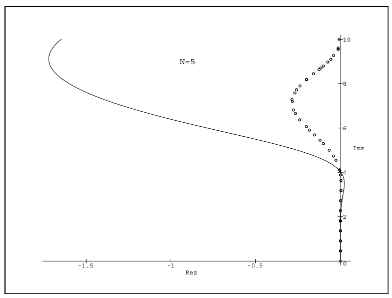

Figure 3. The borders of the stability regions: for uniformly distributed nodes - solid line, for Chebyshev nodes - circles

3. SCHEMES USING ONLY THE LAST ROW OF PSEUDOINVERSE MATRICES

For solving problem (2.1) we reduce again this problem to (2.2) in the standardized interval [-1,1]. Supposing that nodes are

$$-1 \leq x_0 < x_1 < \ldots < x_N < x_{N+1} = 1 \tag{3.1}$$

and denoting

$$p_{N+2}(x) = (x-1)q_{N+1}(x), \quad q_{N+1}(x) = \prod_{k=0}^{N}(x - x_k), \tag{3.2}$$

we can construct the pseudoinverse matrix $\mathbf{B}_{N+2}$ with respect to nodes (3.1). Using only the last row of those matrices elements b_{ik} of which have representations given in [1] we obtain analogously with (2.7) the following computing scheme

$$y_{N+1} - \frac{H}{2} b_{N+1} F(x_{N+1}, y_{N+1}) = y_0 + \frac{H}{2} \sum_{k=0}^{N} b_k F(x_k, y_k), \tag{3.3}$$

$$b_k := b_{N+1,k} = \frac{1}{p'_{N+2}(x_k)}\left[\int_{x_0}^{1}\frac{p_{N+2}(\tau)d\tau}{\tau - x_k} + c\right]. \tag{3.4}$$

In particular,

$$b_{N+1} = \frac{1}{q_{N+1}(1)}\left[\int_{x_0}^{1} q_{N+1}(\tau)d\tau + c\right], \tag{3.5}$$

where c is an arbitrary constant. An application of (3.3) to the problem (2.2) is similar to the (2.7) if we assume again that y_k as the starting values at nodes $x_k,\ \ k = 0, 1, \ldots, N$, are known. Now we will investigate a stability of the method represented by (3.3) and (3.4).

Let us consider the Dalquist test equation in the standardized interval $[-1, 1]$

$$y' = zy, \quad y|_{x=x_0} = y_0, \quad z = \frac{H\lambda}{2} \in \mathbf{C}. \tag{3.6}$$

The precise solution of (3.6) is $y(x) = y_0 \exp[z(x - x_0)]$.

Theorem 3.1. *A stability function composed for the Dalquist test problem* (3.6) *by means of the computing scheme* (3.3), (3.4) *and defined as* $R(z) = y_{N+1}/y_0$ *has the representation*

$$R(z) = \exp[z(1 - x_0)] - \frac{\exp(-zx_0)}{1 - zb_{N+1}}\left[\frac{z^{N+3}}{(N+2)!}\sum_{k=0}^{+\infty}\frac{d_k z^k}{(N+3)_k}\right.$$

$$\left. - \frac{cz^{N+2}}{(N+1)!}\sum_{k=0}^{+\infty}\frac{s_k z^k}{(N+2)_k}\right], \tag{3.7}$$

where

$$d_k = \sum_{j=0}^{k} c_j s_{k-j}, \quad c_j = \int_{x_0}^{1}\tau^j p_{N+2}(\tau)d\tau, \tag{3.8}$$

and s_j *are coefficients in the Laurent series*

$$\frac{1}{p_{N+2}(z)} = z^{-N-2}\sum_{j=0}^{+\infty}\frac{s_j}{z^j}, \quad |z| > 1. \tag{3.9}$$

In particular, $s_0 = 1$. $R(z)$ *is a meromorphic function of the complex variable* z.

Proof. The solution y_{N+1} obtained by (3.3) is

$$(1 - zb_{N+1})\frac{y_{N+1}}{y_0} = 1 + z\sum_{k=0}^{N} b_k \bar{y}(x_k), \tag{3.10}$$

where $\bar{y}(x_k) = (y(x_k))/y_0$ and because of (3.4)

$$b_k = \frac{Q_{N+1}(x_k)}{p'_{N+2}(x_k)}, \quad Q_{N+1}(x_k) = \int_{x_0}^{1} \frac{p_{N+2}(\tau)d\tau}{\tau - x_k} + c. \tag{3.11}$$

Here $Q_{N+1}(s)$ is a polynomial of degree $N+1$ with respect to s. One has the representation

$$Q_{N+1}(s) = \int_{x_0}^{1} \frac{p_{N+2}(\tau) - p_{N+2}(s)}{\tau - s} d\tau + c. \tag{3.12}$$

Now we rewrite (3.10) in the form

$$(1 - zb_{N+1})\frac{y_{N+1}}{y_0} = 1 + z\sum_{k=0}^{N} Res\left[\frac{Q_{N+1}(\tau)\bar{y}(\tau)}{p_{N+2}(\tau)}, x_k\right]. \tag{3.13}$$

Applying the property that the sum of residues in the compactified complex plane is equal to zero we obtain

$$(1 - zb_{N+1})\frac{y_{N+1}}{y_0} = 1 - zRes\left[\frac{Q_{N+1}(\tau)\bar{y}(\tau)}{p_{N+2}(\tau)}, 1\right] - zRes\left[\frac{Q_{N+1}(\tau)\bar{y}(\tau)}{p_{N+2}(\tau)}, \infty\right] \tag{3.14}$$

Substituting $\bar{y}(\tau) = \exp[z(\tau - x_0)]$ we can compute the residues in (3.14):

$$Res\left[\frac{Q_{N+1}(\tau)\exp[z(\tau - x_0)]}{p_{N+2}(\tau)}, 1\right] = \exp[z(1 - x_0)]b_{N+1}, \tag{3.15}$$

$$Res\left[\frac{Q_{N+1}(\tau)\exp[z(\tau - x_0)]}{p_{N+2}(\tau)}, \infty\right] = \exp(-x_0 z)Res\left[\frac{Q_{N+1}(\tau)\exp(z\tau)}{p_{N+2}(\tau)}, \infty\right]. \tag{3.16}$$

Now we will calculate the Laurent series with $|\tau| > 1$ of the function

$$\frac{Q_{N+1}(\tau)}{p_{N+2}(\tau)} = \int_{x_0}^{1} \left[\frac{p_{N+2}(\xi)}{p_{N+2}(\tau)} - 1\right] \frac{d\xi}{\xi - \tau} + \frac{c}{p_{N+2}(\tau)}.$$

Using the expansion $(\xi - \tau)^{-1} = -\sum_{k=0}^{+\infty} \xi^k \tau^{-k-1}$ we obtain

$$\frac{Q_{N+1}(\tau)}{p_{N+2}(\tau)} = \sum_{k=0}^{+\infty} \frac{1 - x_0^{k+1}}{(k+1)\tau^{k+1}} - \tau^{-N-3}\sum_{k=0}^{+\infty} \frac{d_k}{\tau^k} + c\tau^{-N-2}\sum_{k=0}^{+\infty} \frac{s_k}{\tau^k} \tag{3.17}$$

where d_k and s_k are determined by (3.8) and (3.9). Therefore,

$$zRes\left[\frac{Q_{N+1}(\tau)\exp(z\tau)}{p_{N+2}(\tau)}, \infty\right] = -\exp z + \exp(x_0 z) +$$

$$+\frac{z^{N+3}}{(N+2)!}\sum_{k=0}^{+\infty}\frac{d_k z^k}{(N+3)_k}-\frac{cz^{N+2}}{(N+1)!}\sum_{k=0}^{+\infty}\frac{s_k z^k}{(N+2)_k}. \tag{3.18}$$

Using (3.14), (3.15) (3.16) and (3.18) we obtain (3.7). ■

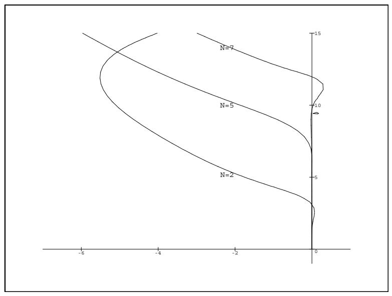

Figure 4. The borders of stability regions for the pseudoinverse matrix with Chebyshev nodes

Corollary 3.1. 1. If $c \neq 0$ the method (3.3) by an arbitrary choice of nodes x_k has the order $N+2$.

2. If $c = 0$ it is possible to increase the order of the method (3.3)- (3.4), specially choosing nodes x_k in $[-1, 1]$ so that the equalities

$$d_m \equiv \sum_{j=0}^{m} c_j s_{m-j} = 0 \quad \text{for} \quad m = 0, 1, \ldots, s. \tag{3.19}$$

are satisfied. Then the order of the method is $N+s+3$. The maximal s can be $N-1$.

3. Let $x_0 = -1$ and $c = 0$. Since $1 = s_0 \neq 0$ the maximum of the order can be reached by (3.8) if

$$\int_{-1}^{1} \tau^k (1-\tau^2) q_N(\tau) d\tau = 0, \quad k = 0, 1, \ldots, N-1. \tag{3.20}$$

It means that the nodes x_k must be taken as zeroes of polynomial Jacobi $P_N^{(1,1)}(x)$ in the interval [-1,1]. In this case the order of the method is $2N+2$.

Example 3.1. There are some samples of the borders of the stability regions for the procedure (3.3), (3.4) with $x_0 = -1$ and $c = 0$ using $R(z)$ defined by (3.7).

1°. In the Fig.4 the borders of stability regions for $N = 2, 5, 7$ nodes as zeroes of Chebyshev polynomials of the second kind are shown .

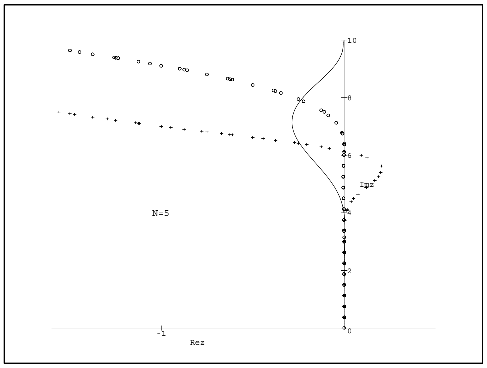

Figure 5. The borders of stability regions: 1) pseudoinverse matrix with uniformly distributed nodes (crosses); 2) pseudoinverse matrix with Chebyshev nodes (circles); 3) matrix for derivatives with Chebyshev nodes (solid line)

2°. In the Fig.5 the borders of stability regions in the case $N = 5$ are compared.

Remark 3.1. Unfortunately, the functions $R(z)$ give only a part of instability domain for the methods because the inequality $|R(z)| \le 1$ in general is not sufficient for the stability.

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DAUGIAŽINGSNIS IŠSIGIMUSIOS MATRICOS METODAS PAPRASTOSIOMS DIFERENCIALINĖMS LYGTIMS

T. Cirulis, D. Cirule, O. Lietuvietis

Pasiūlytas skaitinis metodas paprastosioms diferencialinėms lygtims su pradinėmis sąlygomis spręsti. Išvestinėms aproksimuoti taikoma matrica, kuri nėra reguliarioji. Tai paaiškina metodo pavadinimą. Metodo stabilumui tirti taikoma kompleksinių funkcijų rezidumų teorija. Išnagrinėti atvejai, kai nepriklausomą kintamąjį atitinkantys skaičiavimo schemos mazgai yra paskirstyti intervale kaip tolygiai, taip ir netolygiai.