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DETERMINATION OF THE STABILITY BOUNDARIES FOR THE HAMILTONIAN SYSTEMS WITH PERIODIC COEFFICIENTS[1]

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Abstract. We consider the hamiltonian system of linear differential equations with periodic coefficients. Using the infinite determinant method based on the existence of periodic solutions on the boundaries between the domains of stability and instability in the parameter space we have developed the algorithm for analytical computation of the stability boundaries. The algorithm has been realized for the second and the fourth order hamiltonian systems arising in the restricted many-body problems. The stability boundaries have been found in the form of powers series, accurate to the sixth order in a small parameter. All the computations are done with the computer algebra system *Mathematica*.

Key words: Hamiltonian systems, stability, infinite determinant method, characteristic multipliers

1. Introduction

Let us consider the linear hamiltonian system of differential equations

$$\frac{dx}{dt} = JH(t,\varepsilon)x, \tag{1.1}$$

where $x^T = (x_1, x_2, \ldots, x_{2n})$ ia a $2n$-dimensional vector whose components x_k and x_{n+k} are the canonically conjugated variables, $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ and E_n is the $n \times n$ identity matrix, $H(t,\varepsilon)$ is the real-valued $2n \times 2n$ matrix function which can be represented in the form of the converging series

$$H(t,\varepsilon) = H_0 + \varepsilon H_1(t) + \varepsilon^2 H_2(t) + \ldots, \tag{1.2}$$

[1] The author is very grateful to Prof. Evgenii A. Grebenikov for useful advice and fruitful discussions of the stability problem

where ε is a small parameter. The matrix functions $H_k(t)$ $(k = 1, 2, \ldots)$ in (1.2) are continuous and periodic with a period T, while H_0 is a constant matrix. Besides, H_0 and $H_k(t)$ can depend on some parameters. Equations of the form (1.1) describe dynamical systems with intrinsic periodicity and appear in many branches of science and engineering (see, for example, [11]). Our interest to the system (1.1) arises because such a system occurs in studying the stability of equilibrium solutions in the elliptic restricted many-body problems [3, 9]. We are interested also in determination of the boundaries between the domains of stability and instability in the parameter space for the system (1.1).

According to the general theory of differential equations with periodic coefficients [11], the behaviour of solutions of the system (1.1) is determined by its characteristic exponents which are continuous functions of ε. And the system may be stable only if none of the eigenvalues of the matrix JH_0 has a positive real part. However, the system being stable for $\varepsilon = 0$ may become unstable even for very small values of $\varepsilon > 0$. So, in order to analyse the stability of system (1.1) we have to calculate its characteristic exponents for $\varepsilon > 0$. Using the method of a small parameter, we can find them in the form of power series in ε as it was done in [3, 8], for example. But if we are looking for the stability boundaries the method of infinite determinant turns out to be more effective [4, 7]. It was developed first by Bolotin for a restricted class of differential equations, namely, for a system of uncoupled canonical equations [1]. Then Lindh and Likins extended this method for completely damped mechanical systems [6]. But in both cases the stability boundaries were determined numerically.

Now there are modern computer algebra systems such as *Mathematica* [10], for example, that essentially increases our ability in doing symbolic calculations. The main aim of the present paper is to develop the algorithm for analytical calculation of the stability boundaries and to use it for the hamiltonian systems of the second and the fourth order. All calculations are done with the computer algebra system *Mathematica*.

2. Properties of the Characteristic Multipliers

The hamiltonian systems of linear differential equations with periodic coefficients and their general properties have been studied quite well. It is known that their characteristic multipliers obey the Liapunov-Poincare reciprocal root Theorem (see [5, 11]). It indicates that a hamiltonian system with characteristic multiplier ρ of multiplicity m must also have characteristic multiplier ρ^{-1} of the same multiplicity m. Besides, coefficients in the equation (1.1) are real-valued functions. Therefore, we can formulate the following theorem, which restricts substantially possible values of the characteristic multipliers for the hamiltonian systems.

Theorem 1. *If ρ is a characteristic multiplier for the system* (1.1) *then ρ^{-1}, $\overline{\rho}$, $\overline{\rho}^{-1}$ are its characteristic multipliers as well, where $\overline{\rho}$ is a complex-conjugate value for ρ.*

Hence, characteristic multipliers of the hamiltonian system are divided into groups each of which in general case consists of four elements, namely, ρ, ρ^{-1},

$\overline{\rho}$, $\overline{\rho}^{-1}$. If $|\rho| \neq 1$ they are situated in the complex plane symmetrically both with respect to the unit circle and with respect to the real axis (see Fig. 1). Of course, theorem 1 will be satisfied if $\rho = \overline{\rho}$ and two characteristic multipliers are situated on the real axis symmetrically with respect to the unit circle or $|\rho| = 1$ and they are on the unit circle symmetrically with respect to the real axis. But in any case, if $|\rho| \neq 1$ then there exists at least one characteristic multiplier of modulus exceeding unity which means instability of the system. Thus, the hamiltonian system may be stable only if all of its characteristic multipliers are situated on the unit circle in the complex plane.

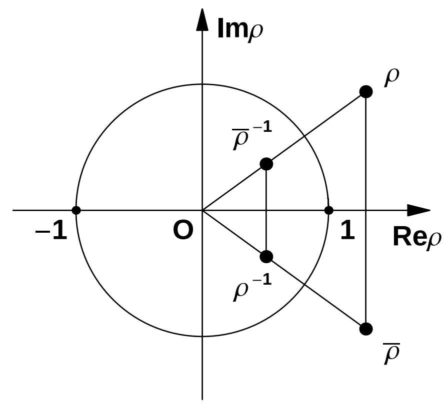

Figure 1. Position of the characteristic multipliers in the complex plane.

Let us suppose that for $\varepsilon = 0$ all characteristic multipliers of the system (1.1) are different complex numbers of unit modulus, i.e., the system is stable. According to Theorem 1, there exist n pairs of complex-conjugate characteristic multipliers situated on the unit circle in the complex plane symmetrically with respect to the real axis. If for $\varepsilon > 0$ modulus of one characteristic multiplier becomes greater than 1, for example, and it leaves the circle then, according to Theorem 1, three additional characteristic multipliers shown on Fig. 1 must arise. But in this case the number of characteristic multipliers will become greater than $2n$ what is impossible. Hence, all characteristic multipliers must stay on the circle and the system will be stable for sufficiently small values of $\varepsilon > 0$. However, if for $\varepsilon = 0$ there exist multiple characteristic multipliers, for example, one pair of complex-conjugate characteristic multipliers has a multiplicity 2, then they can leave the circle without disturbing Theorem 1. Indeed, they can move along radial directions and form a configuration shown on Fig. 1. Thus, the existence of multiple characteristic multipliers is the necessary condition of instability of the hamiltonian system (1.1) for $\varepsilon > 0$. It should be noted also that if there exists at least one characteristic multiplier of the system (1.1) whose modulus is not equal to unity for $\varepsilon = 0$ then the system will be unstable for sufficiently small values of $\varepsilon > 0$ as well because its characteristic multipliers are continuous functions of ε.

3. Computing the Stability Boundaries for the Second Order Hamiltonian System

Let us consider the linear second order differential equation of the form

$$\frac{d^2z}{dt^2} + \frac{a + \varepsilon\cos t}{1 + \varepsilon\cos t}\, z(t) = 0, \tag{3.1}$$

where a, ε are some positive parameters. It arises in studying the stability of equilibrium solutions in the elliptic restricted many-body problems, where the motion of a particle of infinitesimal mass in the gravitational field generated by $(N+1)$ point particles P_0, $P_1, \ldots, P_N$ is investigated [3, 9]. The particles $P_1, \ldots, P_N$ have equal masses and move in elliptic orbits about their common center of mass being at any instant of time in the vertices of a regular polygon with N sides. The polygon rotates about its center where the particle P_0 is resting. It is supposed that orbits of the particles $P_1, \ldots, P_N$ are situated in the xOy plane of the barycentric inertial frame of reference and its origin is at the point P_0. Then equation (3.1) describes the disturbed motion of the particle along Oz axis.

Obviously, equation (3.1) can be written in the form of the second order hamiltonian system (1.1) with the matrix function

$$H(t, \varepsilon) = \begin{pmatrix} \dfrac{a + \varepsilon\cos t}{1 + \varepsilon\cos t} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

The matrix function (3.2) is periodic with the period $T = 2\pi$ and can be represented in the form (1.2), where

$$H_0 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad H_k(t) = (-\cos t)^k \begin{pmatrix} a-1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the corresponding series converges in the domain $|\varepsilon| < 1$ for any t.

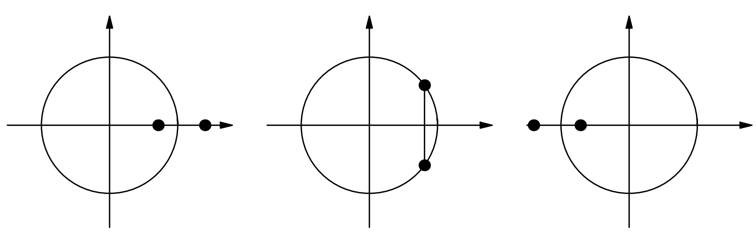

Figure 2. Characteristic multipliers of the second order system.

The second order hamiltonian system has two characteristic multipliers ρ_1 and ρ_2 which must satisfy Theorem 1. Hence, they must be situated in the complex plane either on the real axis or on the unit circle (see Fig. 2). The first case corresponds to unstable behaviour of the system because one of its characteristic multipliers has modulus exceeding unity. In the second case $|\rho_1| = |\rho_2| = 1$ and $\rho_2 = \overline{\rho}_1$ and the system is stable. Changing parameters of the system, we can force characteristic multipliers to move in the complex plane. But its transition from stable to unstable behaviour and back is possible only via the points $\rho = \pm 1$ where the system has multiple characteristic multipliers. Thus, the cases $\rho_1 = \rho_2 = 1$ and $\rho_1 = \rho_2 = -1$ correspond to the boundaries between stable and unstable behaviour of the system.

The eigenvalues of the matrix JH_0 are easily found and can be written as $\lambda_{1,2} = \pm i\sqrt{a}$. The corresponding characteristic multipliers

$$\rho_{1,2} = \exp(2\pi\lambda_{1,2}) = \exp(\pm 2\pi i\sqrt{a})$$

are complex-conjugate numbers of unit modulus. Obviously, the conditions $\rho_1 = \rho_2 = 1$ or $\rho_1 = \rho_2 = -1$ are fulfilled only if

$$a = \frac{k^2}{4} \ \ (k = 1, 2, \ldots). \tag{3.3}$$

Hence, the domains of instability in the $a-\varepsilon$ plane can arise only in the vicinity of the points (3.3). The significant point here is that the cases $\rho_{1,2} = 1$ and $\rho_{1,2} = -1$ are characterized by the existence of periodic solutions of the system (1.1) with periods T and $2T$ respectively. Thus, the boundaries between the domains of stability and instability in the $a - \varepsilon$ plane are some curves $a = a(\varepsilon)$ which are characterized by the presence of periodic solutions with the periods $T = 2\pi$ or $2T = 4\pi$ and which cross the a-axis in the points (3.3).

Now we can attempt to seek a solution of the equation (3.1) in the form of Fourier series

$$z = c_0 + \sum_{k=1}^{\infty} \Big(c_k \ \cos\big(\frac{k}{2}\, t\big) + d_k \ \sin\big(\frac{k}{2}\, t\big)\Big). \tag{3.4}$$

Although this is a Fourier series for the function $z = z(t)$ of period 4π, it can also be used to obtain the solution with period 2π by setting to zero the Fourier coefficients corresponding to k being an odd integer. By substituting (3.4) into equation (3.1) and equating coefficients of $\cos(\frac{k}{2}\, t)$ and $\sin(\frac{k}{2}\, t)$ to zero we obtain the following infinite sequence of equations determining coefficients of the Fourier series (3.4):

$$\begin{cases} a\, c_0 \ = \ 0, \\ \varepsilon\, c_0 + (a-1)\, c_2 - \dfrac{3}{2}\, \varepsilon\, c_4 = 0, \\ \ \vdots \\ -\dfrac{k(k-2)}{2}\, \varepsilon\, c_{2k-2} + (a-k^2)\, c_{2k} - \dfrac{k(k+2)}{2}\, \varepsilon\, c_{2k+2} = 0, \ \ldots \end{cases} \tag{3.5}$$

$$\begin{cases} \big(a - \dfrac{1}{4} + \dfrac{3}{8}\, \varepsilon\big)\, c_1 - \dfrac{5}{8}\, \varepsilon\, c_3 = 0, \\ \ \vdots \\ -\dfrac{(2k-5)(2k-1)}{8}\, \varepsilon\, c_{2k-3} + \Big(a - \big(k - \dfrac{1}{2}\big)^2\Big)\, c_{2k-1} \\ \qquad\qquad - \dfrac{(2k-1)(2k+3)}{8}\, \varepsilon\, c_{2k+1} = 0, \ \ldots \end{cases} \tag{3.6}$$

$$\begin{cases} (a-1)\,d_2 - \dfrac{3}{2}\,\varepsilon\,d_4 = 0, \\ \vdots \\ -\dfrac{k(k-2)}{2}\,\varepsilon\,d_{2k-2} + (a-k^2)\,d_{2k} - \dfrac{k(k+2)}{2}\,\varepsilon\,d_{2k+2} = 0, \ \dots \end{cases} \tag{3.7}$$

$$\begin{cases} \big(a - \dfrac{1}{4} - \dfrac{3}{8}\,\varepsilon\big)\,d_1 - \dfrac{5}{8}\,\varepsilon\,d_3 = 0, \\ \vdots \\ -\dfrac{(2k-5)(2k-1)}{8}\,\varepsilon\,d_{2k-3} + \big(a-(k-\frac{1}{2})^2\big)\,d_{2k-1} \\ \qquad\qquad -\dfrac{(2k-1)(2k+3)}{8}\,\varepsilon\,d_{2k+1} = 0, \ \dots \end{cases} \tag{3.8}$$

It can be seen that in fact there are four infinite subsequences of linear homogeneous equations (3.5) – (3.8). Systems (3.5) and (3.7) are for coefficients c_0, $c_2, \dots , c_{2k}$ and $d_2, \dots , d_{2k}$ respectively and represent solution (3.4) with period 2π. For a solution to exist, the corresponding determinants of infinite systems (3.5), (3.7) must vanish, thus determining the stability boundaries in the $a-\varepsilon$ plane. These boundaries obviously reduce to $a = k^2$ $(k = 0, 1, 2, \dots)$ when $\varepsilon \to 0$. The remaining two subsequences of equations (3.6) and (3.8) are for coefficients $c_1, c_3, \dots , c_{2k+1}$ and $d_1, \dots , d_{2k+1}$ and they correspond to those stability boundaries which reduce to $a = \frac{(2k-1)^2}{4}$ $(k = 1, 2, \dots)$ when $\varepsilon \to 0$.

Of course, it's impossible to calculate a determinant of the infinite matrix. So, in order to find the stability boundaries $a = a(\varepsilon)$ we should truncate the infinite subsequences of equations (3.5) – (3.8) after the k-th term, where k is a suitably large number. The corresponding determinant for the system (3.5), for instance, can be written as

$$D_k = \begin{vmatrix} a & 0 & 0 & 0 & \dots & 0 \\ \varepsilon & a-1 & -\frac{3}{2}\,\varepsilon & 0 & \dots & 0 \\ 0 & 0 & a-4 & -4\varepsilon & \dots & 0 \\ 0 & 0 & -\frac{3}{2}\,\varepsilon & a-9 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -\frac{k(k-2)}{2}\,\varepsilon & a-k^2 \end{vmatrix}. \tag{3.9}$$

Equating determinant (3.9) to zero we obtain an algebraic equation giving an approximation for the stability boundary $a = a(\varepsilon)$. An exact expression for the boundary is obtained when $k \to \infty$. This approach giving the equation for the stability boundary is known as the infinite determinant method [6].

Determinant (3.9) is most efficiently evaluated from the following recurrence relation

$$D_k = (a-k^2)D_{k-1} - \frac{\varepsilon^2}{2}\,(k-2)(k-1)k(k+1)D_{k-2}, \quad k \geq 3, \tag{3.10}$$

which is readily established from (3.9). To start the iterative process we observe that

$$D_1 = a, \quad D_2 = a(a-1).$$

A similar procedure can be followed for the other systems (3.6) – (3.8). For instance, determinant of the system (3.7) is just the same as (3.9) with the first row and column deleted. The recurrence relation is again (3.10) for $k \geq 3$, but the starting values are now given by

$$D_1 = a-1, \quad D_2 = (a-1)(a-4).$$

The corresponding determinants for the system (3.6), (3.8) are

$$D_k = \begin{vmatrix} a-\frac{1}{4} \pm \frac{3}{8}\,\varepsilon & -\frac{5}{8}\,\varepsilon & 0 & \dots & 0 \\ \frac{3}{8}\,\varepsilon & a-\frac{9}{4} & -\frac{21}{8}\,\varepsilon & \dots & 0 \\ 0 & -\frac{5}{8}\,\varepsilon & a-\frac{25}{4} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -\frac{(2k-1)(2k-5)}{8}\,\varepsilon & a-\frac{(2k-1)^2}{4} \end{vmatrix}. \tag{3.11}$$

The recurrence relation for the determinants (3.11) is

$$D_k = (a - \frac{(2k-1)^2}{4})\, D_{k-1} - \frac{\varepsilon^2}{64}\,(2k-5)(2k-3)(2k-1)(2k+1)\, D_{k-2} \tag{3.12}$$

with the starting values

$$D_1 = a - \frac{1}{4} \pm \frac{3\varepsilon}{8}, \quad D_2 = (a - \frac{1}{4} \pm \frac{3\varepsilon}{8})(a - \frac{9}{4}) + \frac{15\varepsilon^2}{64}.$$

It is evident from (3.10), (3.12) that in the case of $\varepsilon = 0$ determinants of systems (3.5) – (3.8) will be equal to zero when $a = \frac{1}{4}\,k^2$ $(k = 0, 1, 2, \ldots)$. It means that the stability boundaries cross the $\varepsilon = 0$ axis in the $a - \varepsilon$ plane at the points (3.3). For sufficiently small ε we can represent the corresponding curves $a = a(\varepsilon)$ in the vicinity of these points as power series

$$a = \frac{k^2}{4} + a_1\varepsilon + a_2\varepsilon^2 + \ \dots \quad (k = 0, 1, 2 \dots). \tag{3.13}$$

It is easy to show from (3.10), (3.12) that in order to find the curves (3.13) in the vicinity of the point $a = \frac{1}{4}k^2$ with accuracy $o(\varepsilon^{2n})$, it is sufficient to calculate the determinant D_{k+n}. Then we should substitute (3.13) into the expression for D_{k+n} and expand it in powers of ε. Afterwards, equating coefficients of ε^k $(k = 1, 2, 3, \dots)$ to zero, we obtain a system of algebraic equations determining the coefficients a_k in the expansion (3.13). As a result we have found the following curves

$$a = 0, \;\; a = 1, \;\; a = 4 - \frac{6}{5}\,\varepsilon^2 - \frac{39}{125}\,\varepsilon^4 - \frac{7023}{43750}\,\varepsilon^6,$$

$$a = 9 - \frac{108}{35}\,\varepsilon^2 - \frac{139617}{171500}\,\varepsilon^4 - \frac{177754233}{420175000}\,\varepsilon^6,$$
$$a = 16 - \frac{40}{7}\varepsilon^2 - \frac{4670}{3087}\,\varepsilon^4 - \frac{11800015}{14975037}\,\varepsilon^6,$$
$$a = 25 - \frac{100}{11}\,\varepsilon^2 - \frac{115475}{47916}\,\varepsilon^4 - \frac{11932109425}{9496855368}\,\varepsilon^6, \tag{3.14}$$

$$a = \frac{1}{4} \mp \frac{3}{8}\,\varepsilon + \frac{15}{128}\,\varepsilon^2 \mp \frac{45}{2048}\,\varepsilon^3 + \frac{885}{32768}\,\varepsilon^4 \mp \frac{6105}{524288}\,\varepsilon^5 + \frac{220305}{16777216}\,\varepsilon^6,$$
$$a = \frac{9}{4} - \frac{135}{256}\,\varepsilon^2 \mp \frac{45}{2048}\,\varepsilon^3 - \frac{34695}{262144}\,\varepsilon^4 \mp \frac{23895}{2097152}\,\varepsilon^5 - \frac{8975205}{134217728}\,\varepsilon^6,$$
$$a = \frac{25}{4} - \frac{525}{256}\,\varepsilon^2 - \frac{141225}{262144}\,\varepsilon^4 \mp \frac{525}{2097152}\,\varepsilon^5 - \frac{37465575}{134217728}\,\varepsilon^6,$$
$$a = \frac{49}{4} - \frac{2205}{512}\,\varepsilon^2 - \frac{2388015}{2097152}\,\varepsilon^4 - \frac{2545233705}{4294967296}\,\varepsilon^6. \tag{3.15}$$

We see that systems (3.5), (3.7) determine two different curves (3.15) crossing the $\varepsilon = 0$ axis at the points $a = \frac{(2k-1)^2}{4}$ $(k = 1, 2, \dots)$, while systems (3.6), (3.8) determine the same curves (3.14). Hence, the domains of instability for the equation (1.1) with matrix function (3.2) exist only between the curves (3.15). Thus, we can formulate the following theorem.

Theorem 2. *The domains of instability for the second order hamiltonian system* (1.1) *with the matrix function* (3.2) *exist only in the vicinity of the points*

$$a = \frac{(2k-1)^2}{4} \quad (k = 1, 2, \dots)$$

in the $a - \varepsilon$ *plane and are bounded by the curves* (3.15)*. The bandwidth of these domains is* $O(\varepsilon^{2k-1})$ *and decreases very fast if the number* k *is growing up.*

It should be emphasized that, increasing the order n of determinant D_n of the systems (3.5) – (3.7), we'll be able to find only small correction terms in the equations of the stability boundaries (3.13). The terms we have already found in (3.14), (3.15) will be the same.

4. Hamiltonian System of the Fourth Order

4.1. Characteristic multipliers for the fourth order system

The fourth order hamiltonian system has four characteristic multipliers which must obey Theorem 1 as well. Hence, the system may be stable only if all its characteristic multipliers are complex-valued with unit magnitude

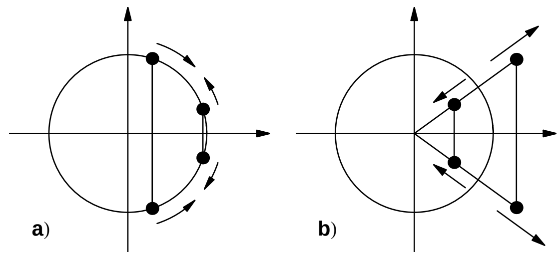

Figure 3. Motion of the characteristic multipliers in the complex plane.

$$|\rho_1| = |\rho_2| = |\rho_3| = |\rho_4| = 1.$$

Geometrically these characteristic multipliers are situated on the unit circle in the complex plane, symmetrically in pairs with respect to the real axis (see Fig. 3a).

System (1.1) becomes unstable if at least one characteristic multiplier leaves the circle. But Theorem 1 imposes restrictions on possible motion of the characteristic multipliers in the complex plane. One possibility is shown in Fig. 3b, when two characteristic multipliers, being in the same semi-plane, move toward each other on the circle until their coincidence and then start to move away of the circle along radial directions. It means that the system becomes unstable because modulus of two characteristic multipliers becomes greater than 1. If such a case is realized then system (1.1) has no periodic solutions and we have to calculate its characteristic multipliers explicitly in order to find the stability boundaries. There is also another possibility when two characteristic multipliers, moving on the circle toward each other, coincide in the point $\rho = -1$ and then continue their motion along the real axis (see Fig. 4).

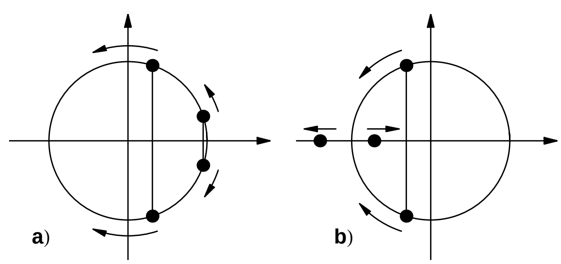

Figure 4. Transition of two characteristic multipliers into the real axis.

Similar situation can occur if two characteristic multipliers coincide in the point $\rho = 1$. In both cases the other two characteristic multipliers remain on the unit circle, being symmetrical with respect to the real axis. Again the cases $\rho = \pm 1$ correspond to the boundary between stable and unstable behaviour of the system (1.1), similarly as it is in the case of the second order hamiltonian system. The interesting and significant result from this analysis is that for $\rho = \pm 1$ the system (1.1) has periodic solutions of period T and $2T$ respectively and this property may be used for determination of the boundaries between the domains of stability and instability in the parameter space.

4.2. Computing the stability boundaries

Let us consider now the hamiltonian system (1.1) of the fourth order with the matrix function

$$H(t,\varepsilon) = \begin{pmatrix} \dfrac{1+b+4\varepsilon\cos t}{1+\varepsilon\cos t} & 0 & 0 & -2 \\ 0 & -\dfrac{b}{1+\varepsilon\cos t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

where b and ε are some positive parameters. Such a system describes the disturbed motion of the particle in the xOy plane in the elliptic restricted problem of four bodies [2]. The matrix function (4.1) is periodic with the period $T = 2\pi$ and may be represented in the form (1.2), where

$$H_0 = \begin{pmatrix} 1+b & 0 & 0 & -2 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad H_k(t) = (-\cos t)^k \begin{pmatrix} -3+b & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the corresponding series converges in the domain $|\varepsilon| < 1$ for any t. Hence, characteristic exponents for the system are continuous functions of ε. In the case of $\varepsilon = 0$ they are just the eigenvalues of the matrix JH_0 and can be represented as $\lambda_{1,2} = \pm i\sigma_1$, $\lambda_{3,4} = \pm i\sigma_2$ where

$$\sigma_{1,2} = \left(\frac{1 \pm \sqrt{1-12b+4b^2}}{2}\right)^{1/2}.$$

They are distinct pure imaginary numbers if parameter b satisfies the following inequalities

$$0 < b < \frac{1}{4}(6-\sqrt{32}) \quad or \quad \frac{1}{4}(6+\sqrt{32}) < b < 3. \tag{4.2}$$

Thus, the considered hamiltonian system may be stable for $\varepsilon > 0$ only if parameter b belongs to the intervals (4.2). In other cases there exists at least one characteristic exponent with a positive real part and the system will be unstable for sufficiently small ε.

Let us consider the first interval in (4.2). The corresponding intervals for $\sigma_{1,2}$ can be easily found and are given by

$$\frac{1}{\sqrt{2}} < \sigma_1 < 1, \ \ 0 < \sigma_2 < \frac{1}{\sqrt{2}}.$$

Obviously, there is only one possibility for the system to have multiple characteristic multipliers. It is just the case $\sigma_2 = \frac{1}{2}$ when two characteristic multipliers

$$\rho_{3,4} = \exp(\pm 2\pi\sigma_2 i) = -1.$$

The corresponding geometrical configuration is shown in Fig. 4. From the analysis above it follows that the domain of instability can arise only in the vicinity of the point

$$b = \frac{1}{4}(6 - \sqrt{33}), \tag{4.3}$$

for which $\sigma_2 = \frac{1}{2}$. Hence, the boundaries between the domains of stability and instability in the $b - \varepsilon$ plane are some curves $b = b(\varepsilon)$ which are characterized by the presence of periodic solutions with the period $2T = 4\pi$ and cross the b-axis at the point (4.3).

In order to find the stability boundaries let us rewrite the system (1.1) with matrix (4.1) in the form of two linear second order differential equations

$$\begin{cases} (1 + \varepsilon\cos t)(\ddot{x}_1 - 2\dot{x}_2) + (-3 + b)x_1 = 0, \\ (1 + \varepsilon\cos t)(\ddot{x}_2 + 2\dot{x}_1) - b\, x_2 = 0, \end{cases} \tag{4.4}$$

where a dot means the derivative $\frac{d}{dt}$. Now we can attempt to seek a solution of the system (4.4) in the form of Fourier series

$$x_1 = p_0 + \sum_{k=1}^{\infty} \big(p_k\ \cos(\frac{k}{2}\ t) + q_k\ \sin(\frac{k}{2}\ t)\big), \tag{4.5}$$

$$x_2 = \alpha_0 + \sum_{k=1}^{\infty} \big(\alpha_k\ \cos(\frac{k}{2}\ t) + \beta_k\ \sin(\frac{k}{2}\ t)\big).$$

By substituting (4.5) into equations (4.4) and equating coefficients of $\cos(\frac{k}{2}\ t)$ and $\sin(\frac{k}{2}\ t)$ to zero we obtain two infinite sequences of linear homogeneous equations. The first system is for the odd coefficients $p_1, p_3,\ \ldots,\ p_{2k-1}$ and $\beta_1,\ \ldots,\ \beta_{2k-1}$ and is given by

$$(-b + \frac{13}{4} + \frac{\varepsilon}{8})\, p_1 + (1 + \frac{\varepsilon}{2})\, \beta_1 + \frac{9\varepsilon}{8}\, p_3 + \frac{3\varepsilon}{2}\, \beta_3 = 0,$$

$$(-1 + \frac{\varepsilon}{2})\, p_1 + (-b - \frac{1}{4} + \frac{\varepsilon}{8})\, \beta_1 - \frac{3\varepsilon}{2}\, p_3 - \frac{9\varepsilon}{8}\, \beta_3 = 0,$$

$$\vdots$$

$$\begin{aligned} &\frac{\varepsilon}{8}(3-2k)^2 p_{2k-3} + \frac{\varepsilon}{2}(-3+2k)\beta_{2k-3} + (-b + \frac{13}{4} - k + k^2)\, p_{2k-1} \\ &\qquad + (-1+2k)\, \beta_{2k-1} + \frac{\varepsilon}{8}(1+2k)^2\, p_{2k+1} + \frac{\varepsilon}{2}(1+2k)\, \beta_{2k+1} = 0, \\ &\frac{\varepsilon}{2}(-3+2k) p_{2k-3} + \frac{\varepsilon}{8}(-3+2k)^2 \beta_{2k-3} + (1-2k)\, p_{2k-1} \\ &\qquad + (-b - \frac{1}{4} + k - k^2)\, \beta_{2k-1} + \frac{\varepsilon}{2}(1+2k)\, p_{2k+1} + \frac{\varepsilon}{8}(1+2k)^2\, \beta_{2k+1} = 0,\ \ldots \end{aligned} \tag{4.6}$$

The second system of equations determines the coefficients $q_1,\ q_3,\ \ldots,\ q_{2k-1}$ and $\alpha_1,\ \ldots,\ \alpha_{2k-1}$ and it can be written as

$$\begin{aligned}
&(-b+\frac{13}{4}-\frac{\varepsilon}{8})\ q_1+(-1+\frac{\varepsilon}{2})\ \alpha_1+\frac{9\varepsilon}{8}\ q_3-\frac{3\varepsilon}{2}\ \alpha_3=0,\\
&(1+\frac{\varepsilon}{2})\ q_1-(b+\frac{1}{4}+\frac{\varepsilon}{8})\ \alpha_1+\frac{3\varepsilon}{2}\ q_3-\frac{9\varepsilon}{8}\ \alpha_3=0,\\
&\qquad\vdots\\
&\frac{\varepsilon}{8}(3-2k)^2 q_{2k-3}-\frac{\varepsilon}{2}(-3+2k)\alpha_{2k-3}+(-b+\frac{13}{4}-k+k^2)\ q_{2k-1}\\
&\qquad+(1-2k)\ \alpha_{2k-1}+\frac{\varepsilon}{8}(1+2k)^2\ q_{2k+1}-\frac{\varepsilon}{2}(1+2k)\ \alpha_{2k+1}=0,\\
&\frac{\varepsilon}{2}(-3+2k)q_{2k-3}-\frac{\varepsilon}{8}(-3+2k)^2\alpha_{2k-3}+(-1+2k)\ q_{2k-1}\\
&\qquad+(-b-\frac{1}{4}+k-k^2)\ \alpha_{2k-1}+\frac{\varepsilon}{2}(1+2k)\ q_{2k+1}-\frac{\varepsilon}{8}(1+2k)^2\ \alpha_{2k+1}=0,\ \ldots
\end{aligned}\tag{4.7}$$

Extracting coefficients of $\cos(k\ t)$ and $\sin(k\ t)$ we can easily obtain two similar sequences of equations for the even coefficients $p_{2k},\ q_{2k},\ \alpha_{2k},\ \beta_{2k}$. For a solution to exist, the corresponding determinants of infinite systems (4.6), (4.7) must vanish, thus determining the stability boundaries in the $b-\varepsilon$ plane. These boundaries obviously must reduce to the point $b=(6-\sqrt{33})/4$ when $\varepsilon\to\ 0$.

In order to find the stability boundaries $b=b(\varepsilon)$ we should truncate the infinite sequences of equations (4.6) – (4.7) after the k-th term, where k is a suitably large number. For $k=3$, for example, the corresponding determinants may be written as

$$D_3=\begin{vmatrix}
-b+\frac{13}{4}\pm\frac{\varepsilon}{8} & \pm1+\frac{\varepsilon}{2} & \frac{9\varepsilon}{8} & \pm\frac{3\varepsilon}{2} & 0 & 0\\
\mp1+\frac{\varepsilon}{2} & -b-\frac{1}{4}\pm\frac{\varepsilon}{8} & \mp\frac{3\varepsilon}{2} & -\frac{9\varepsilon}{8} & 0 & 0\\
\frac{\varepsilon}{8} & \pm\frac{\varepsilon}{2} & -b+\frac{21}{4} & \pm3 & \frac{25\varepsilon}{8} & \pm\frac{5\varepsilon}{2}\\
\mp\frac{\varepsilon}{2} & -\frac{\varepsilon}{8} & \mp3 & -b-\frac{9}{4} & \mp\frac{5\varepsilon}{2} & -\frac{25\varepsilon}{2}\\
0 & 0 & \frac{9\varepsilon}{8} & \pm\frac{3\varepsilon}{2} & -b+\frac{37}{4} & \pm5\\
0 & 0 & \mp\frac{3\varepsilon}{2} & -\frac{9\varepsilon}{8} & \mp5 & -b-\frac{25}{4}
\end{vmatrix}.\tag{4.8}$$

Equating determinants (4.8) to zero we obtain two algebraic equations giving an approximation for $b=b(\varepsilon)$. For sufficiently small ε we can represent the functions $b=b(\varepsilon)$ in the vicinity of the point (4.3) as power series

$$b=\frac{6-\sqrt{33}}{4}+b_1\varepsilon+b_2\varepsilon^2+\ \ldots.\tag{4.9}$$

Substituting (4.9) into (4.8) and equating coefficients of ε^k to zero we get the system of algebraic equations determining the coefficients b_k $(k=1,2,\ldots)$. Solving this system we obtain the stability boundaries in the form

$$\begin{aligned}
b=\frac{6-\sqrt{33}}{4}\pm\frac{\varepsilon}{8}+\frac{23\varepsilon^2}{128\sqrt{33}}\mp\frac{105\varepsilon^3}{2048}-\frac{148859\varepsilon^4}{1081344\sqrt{33}}\\
\mp\frac{58335\varepsilon^5}{5767168}-\frac{1085089447\varepsilon^6}{18270388224\sqrt{33}}\ ,
\end{aligned}\tag{4.10}$$

where the error term is $O(\varepsilon^7)$.

The second interval in (4.2) may be analysed similarly. As a result we obtain the boundaries of the second domain of instability as

$$b = \frac{6+\sqrt{33}}{4} \pm \frac{\varepsilon}{8} - \frac{23\varepsilon^2}{128\sqrt{33}} \mp \frac{105\varepsilon^3}{2048} + \frac{148859\varepsilon^4}{1081344\sqrt{33}} \mp \frac{58335\varepsilon^5}{5767168} + \frac{1085089447\varepsilon^6}{18270388224\sqrt{33}}\,. \tag{4.11}$$

Thus, we can formulate the following theorem.

Theorem 3. *The fourth order hamiltonian system* (1.1) *with the matrix function* (4.1) *may be stable only if parameter* b *belongs to the intervals* (4.2)*. For sufficiently small* $\varepsilon > 0$ *there exist the domains of instability in the vicinity of the points* $b = (6 \pm \sqrt{33})/4$ *in the* $b - \varepsilon$ *plane which are bounded by the curves* (4.10)*,* (4.11)*.*

It should be emphasized that increasing the order n of the determinants D_n of the systems (4.6) – (4.7) we'll be able to find only the higher order coefficients $b_7,\ b_8,\ \ldots$ in the equations of the stability boundaries (4.9). The coefficients $b_1,\ b_2,\ \ldots,\ b_6$ are the same as in (4.10), (4.11).

5. Conclusions

Using the infinite determinant method based on the existence of periodic solutions on the boundaries between the domains of stability and instability in the parameter space we have developed the algorithm for analytical computation of the stability boundaries for the hamiltonian systems of linear differential equations with periodic coefficients. The algorithm has been implemented in the case of the second and the fourth order hamiltonian systems arising in the elliptic restricted many-body problems. The obtained results are in a good agreement with similar results of [2, 3], where the calculations are done with smaller accuracy and another method is used.

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Hamiltono sistemų su periodiniais koeficientais stabilumo tyrimas begalinių determinantų metodu

A.N. Prokopenya

Nagrinėjama Hamiltono tiesinių diferencialinių lygčių su periodiniais koeficientais sistema. Remiantis tuo, kad parametrų erdvėje stabilumo ir nestabilumo sritis skiriančioje sienoje egzistuoja periodinis sprendinys, sukurtas analitinis minėtos sienos apskaičiavimo algoritmas. Algoritmas realizuotas antros ir ketvirtos eilės Hamiltono sistemoms, kylančioms nagrinėjant apribotų keleto kūnų uždavinius. Stabilumo srities siena randama laipsninės eilutės pavidalu mažojo parametro šešto laipsnio tikslumu. Skaičiavimai atlikti skaičiavimo algebros paketo *Mathematica* pagalba.