Mathematical Modelling and Analysis
Volume 11 Number 2, 2006, pages 161–172
 ISSN 1392-6292 print, ISSN 1648-3510 online

NUMERICAL ANALYSIS OF THE WAVE PROPAGATION IN A COMPOSITE MEDIUM

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Received October 10, 2005; revised March 17, 2006; published online May 25, 2006
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Abstract. We consider the wave propagation process in a 2-dimensional material structure composed of random oriented orthotropic crystals and analyse numerical results for the diffraction problem and the problem of waves interaction with a free boundary.

Key words: orthotropic crystals, wave propagation, diffraction, numerical algorithms

1. Introduction

Modelling of short wave propagation process is of key importance for solution of very different problems [2]. We mention only identification and recognition of defects in solids [1, 3]. In paper [2] a model of a linearly elastic composite medium consisting of a matrix containing a set of orthotropic crystals with randomly oriented anisotropy axes was proposed and some numerical results for the wave interaction with a free boundary were given.

In the present paper we use the model given in [2] and examine the waves diffraction and waves interaction with a free boundary. Since equations of the model are of the hyperbolic type there may occur waves caused by the discretization in the computer modelling. As a test problem we examine the one-dimensional model. We give the explicit formula for the solution of this problem and solve it numerically.

The paper is organized as follows. In Section 2 the one-dimensional case is considered. In Section 3 the formulation of the problems in two-dimensional case is given. Numerical results are presented in Section 4. Section 5 concludes the paper.

2. The One-Dimensional Problem

The interaction of waves with a free boundary in one-dimensional case can be described by the model

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = b^2 \dfrac{\partial^2 u}{\partial x^2}, \quad x \in (0,1),\ t > 0, \\ \dfrac{\partial u}{\partial x}\Big|_{x=0} = \begin{cases} -\mu(t), & t \le T, \\ 0, & t > T, \end{cases} \\ \dfrac{\partial u}{\partial x}\Big|_{x=1} = 0, \\ u\big|_{t=0} = \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0. \end{cases} \tag{2.1}$$

Here we have the following coefficients

$$T \le \frac{1}{b},\ \ \mu \in C^1[0,T],\ \ \mu(0) = \mu'(0) = \mu(T) = \mu'(T) = 0,\ \ \int_0^T \mu(\tau)\, d\tau = 0,$$

where the prime indicates differentiation. Define $f(t) = b\int_0^t \mu(\tau)d\tau$. The solution of problem (2.1) is expressed by the formulas (see Fig. 1):

$$u(t,x) = 0 \quad \text{in } D_0 = \left\{0 \le t \le \frac{x}{b}\right\},$$

$$u(t,x) = f\left(t - \frac{x}{b}\right) \quad \text{in } D_1 = \left\{\frac{x}{b} \le t \le \min\left(T + \frac{x}{b}, \frac{2-x}{b}\right)\right\},$$

$$u(t,x) = f\left(t - \frac{x}{b}\right) + f\left(t + \frac{x-2}{b}\right) \quad \text{in } D_2 = \left\{\frac{2-x}{b} \le t \le T + \frac{x}{b}\right\},$$

$$u(t,x) = 0 \quad \text{in } D_3 = \left\{T + \frac{x}{b} \le t \le \frac{2-x}{b}\right\},\ u(t,x) = f\left(t + \frac{x-2}{b}\right) \text{ in } D_4,$$

$$D_4 = \left\{\max\left(\frac{2-x}{b}, T + \frac{x}{b}\right) \le t \le \min\left(T + \frac{2-x}{b}, \frac{2+x}{b}\right)\right\},$$

$$u(t,x) = f\left(t + \frac{x-2}{b}\right) + f\left(t - \frac{2+x}{b}\right) \text{ in } D_5 = \left\{\frac{2+x}{b} \le t \le T + \frac{2-x}{b}\right\},$$

and so on.

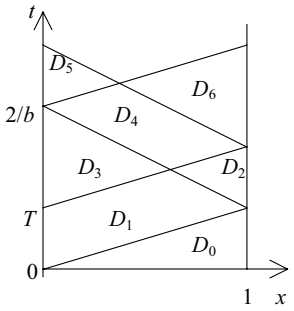

Figure 1. Sketch of the 1D domain for problem (2.1).

The diffraction problem is described by the model

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = b^2 \dfrac{\partial^2 u}{\partial x^2}, \quad x \in (0,1) \cup (1,d),\ t > 0, \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \begin{cases} -\mu(t), & t \le T, \\ 0, & t > T, \end{cases} \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=d} = 0, \\ \left[u\big|_{x=1}\right] = \left[b^2 \dfrac{\partial u}{\partial x}\Big|_{x=1}\right] = 0, \\ u\big|_{t=0} = \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0. \end{cases} \tag{2.2}$$

Here the constant $b = b_1$ if $x \in [0,1)$, $b = b_2$ if $x \in (1,d]$, μ is the same as in model (2.1) and $\left[u\big|_{x=1}\right]$ denotes a jump of u at the point of discontinuity $x = 1$. The solution of problem (2.2) can be expressed by formulas (see Fig. 2):

$$u(t,x) = 0 \quad \text{in } D_0 = \Big\{ t \le \frac{x}{b_1} \text{ for } x \le 1 \text{ and } t \le \frac{1}{b_1} + \frac{x-1}{b_2} \text{ for } x \ge 1 \Big\},$$

$$u(t,x) = f\Big(t - \frac{x}{b_1}\Big) \quad \text{in } D_1 = \Big\{ \frac{x}{b_1} \le t \le \min\Big(\frac{2-x}{b_1}, T + \frac{x}{b_1}\Big),\ x \le 1 \Big\},$$

$$u(t,x) = f\Big(t - \frac{x}{b_1}\Big) + \frac{b_1 - b_2}{b_1 + b_2} f\Big(t + \frac{x-2}{b_1}\Big) \quad \text{in } D_2,$$

$$D_2 = \Big\{ \Big(\frac{2-x}{b_1} \le t \le T + \frac{x}{b_1}\Big),\ x \le 1 \Big\},$$

$$u(t,x) = \frac{2b}{b_1 + b_2} f\Big(t + \frac{1-x}{b_2} - \frac{1}{b_1}\Big) \quad \text{in } D_3,$$

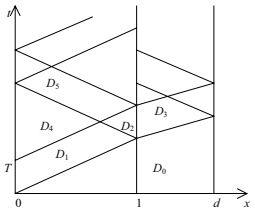

Figure 2. Sketch of the 1D domain for diffraction problem (2.2).

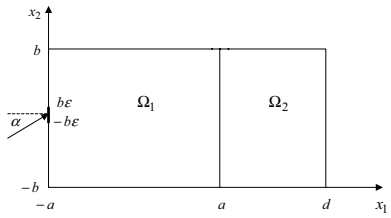

Figure 3. Sketch for the 2D domain of problem (3.1).

$$D_3 = \Big\{ \frac{1}{b_1} + \frac{x-1}{b_2} \le t \le \min\Big(T + \frac{1}{b_1} + \frac{x-1}{b_2}, \frac{1}{b_1} + \frac{2d-1-x}{b_2}\Big), x \ge 1 \Big\},$$

$$u(t,x) = 0, \quad \text{in } D_4 = \Big\{ T + \frac{x}{b_1} \le t \le \frac{2-x}{b_1}, x \le 1 \Big\},$$

$$u(t,x) = \frac{b_1 - b_2}{b_1 + b_2} f\Big(t + \frac{x-2}{b_1}\Big), \quad \text{in } D_5,$$

$$D_5 = \Big\{ \max\Big(T + \frac{x}{b_1}, \frac{2-x}{b_1}\Big), \min\Big(\frac{2+x}{b_1}, T + \frac{2-x}{b_1}\Big), \, x \le 1 \Big\},$$

and so on. In what follows we use the function $\mu(t) = -q \sin^3 \omega t$, $q = const$, $T = 2\pi/\omega$.

3. The 2-Dimensional Problem

Let us first examine the waves diffraction problem i.e. when waves propagate in the rectangular composite plate $\Omega = \Omega_1 + \Omega_2$ (see Fig. 3) with piece-wise constant elasticity characteristics. Waves are generated by the given normal and tangent strains on a small part $(-b\varepsilon, b\varepsilon)$ of one side. The other sides are assumed to be free and the displacement vector u and strains P_{11} and P_{12} have to be continuous at the discontinuity line $x_1 = a$. The model consists of the equations

$$\begin{cases} \rho \dfrac{\partial^2 u_1}{\partial t^2} = A_{1111} \dfrac{\partial^2 u_1}{\partial x_1^2} + A_{1212} \dfrac{\partial^2 u_1}{\partial x_2^2} + (A_{2211} + A_{1212}) \dfrac{\partial^2 u_2}{\partial x_1 \partial x_2}, \\ \rho \dfrac{\partial^2 u_2}{\partial t^2} = (A_{2211} + A_{1212}) \dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{1212} \dfrac{\partial^2 u_2}{\partial x_1^2} + A_{2222} \dfrac{\partial^2 u_2}{\partial x_2^2}, \end{cases} \tag{3.1}$$

$t > 0$, $(x_1, x_2) \in \Omega$ subject to the conditions

$$A_{1111}\frac{\partial u_1}{\partial x_1}+A_{2211}\frac{\partial u_2}{\partial x_2}=\begin{cases}-B(t,x_2)\cos\alpha,\ x_1=-a,\ x_2=[-b\varepsilon,b\varepsilon],\ t\le 1,\\ 0,\quad x_1=-a,\ x_2\in[-b\varepsilon,b\varepsilon],\ t>1,\\ 0,\quad x_1=-a,\ x_2\in[-b,b]\setminus[-b\varepsilon,b\varepsilon],\ t>0,\\ 0,\quad x_1=d,\ x_2\in[-b,b],\ t>0,\end{cases}$$

$$A_{1212}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)=\begin{cases}-B(t,x_2)\sin\alpha,\ x_1=-a,\ x_2\in[-b\varepsilon,b\varepsilon],\ t\le 1,\\ 0,\ x_1=-a,\ x_2\in[-b\varepsilon,b\varepsilon],\ t>1,\end{cases}$$

$$\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=0,\quad x\in\partial\Omega\setminus\{x_1=-a,\ x_2\in[-b\varepsilon,b\varepsilon]\},\tag{3.2}$$

$$A_{2211}\frac{\partial u_1}{\partial x_1}+A_{2222}\frac{\partial u_2}{\partial x_2}=0,\quad x_2=\pm b,\ x_1\in(-a,d),$$

$$u_1\big|_{t=0}=u_2\big|_{t=0}=0,\quad (x_1,x_2)\in\Omega,$$

$$\frac{\partial u_1}{\partial t}\Big|_{t=0}=\frac{\partial u_2}{\partial t}\Big|_{t=0}=0,\quad (x_1,x_2)\in\Omega,$$

$$[u_1]=[u_2]=\left[A_{1111}\frac{\partial u_1}{\partial x_1}+A_{2211}\frac{\partial u_2}{\partial x_2}\right]=\left[A_{1212}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\right]=0,$$

$$x_1=a,\ x_2\in(-b,b).$$

Here ρ is the material density,

$$\begin{aligned}A_{1111}&=0.983C_{1111}+2.91\cdot10^{-4}C_{2222}+9.636\cdot10^{-3}(2C_{1122}+4C_{1212}),\\ A_{2222}&=2.91\cdot10^{-4}C_{1111}+0.983C_{2222}+9.636\cdot10^{-3}(2C_{1122}+4C_{1212}),\\ A_{2211}&=9.636\cdot10^{-3}(C_{1111}+C_{2222}-4C_{1212})+0.983C_{1122},\\ A_{1212}&=9.636\cdot10^{-3}(C_{1111}+C_{2222}-2C_{1122})+0.964C_{1212},\end{aligned}\tag{3.3}$$

$$C_{1111}=\frac{E_1}{1-\nu_{21}^2E_1/E_2},\quad C_{2211}=\frac{E_1\nu_{21}}{1-\nu^2{}_{21}E_1/E_2},$$

$$C_{2222}=\frac{E_2}{1-\nu^2{}_{21}E_1/E_2},\quad C_{1212}=G_{12},$$

where G_{12} signifies the shear modulus, E_1 and E_2 mean elasticity moduli, and ν_{21} is the Poisson coefficient. Coefficients ρ, G_{12}, E_1, E_2, and ν_{21} are different in Ω_1 and Ω_2. In what follows we use the function

$$B(t,x_2)=\eta\sin^3\omega t\big((b\varepsilon)^2-x_2^2\big)^2,\ \ \eta=const>0.$$

Formulas (3.3) are obtained from the equations [2]

$$A_{qqnm} = \int_{-\Theta}^{\Theta} f(\theta)\alpha_{in}\alpha_{jm}\alpha_{sq}\alpha_{kq}\, d\theta\, C_{ijks}, \quad q = 1, 2,$$

$$2A_{12nm} = \int_{-\Theta}^{\Theta} f(\theta)\alpha_{in}\alpha_{jm}(\alpha_{s1}\alpha_{k2} + \alpha_{s2}\alpha_{k1})\, d\theta\, C_{ijks},$$

$$\alpha_{11} = \alpha_{22} = \cos\theta, \quad \alpha_{12} = -\alpha_{21} = \sin\theta,$$

$$\alpha_{13} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 0, \quad \alpha_{33} = 1, \quad \theta = \frac{\pi}{4},$$

$$f(\theta) = e^{-\theta^2/2\sigma^2}\left(2\int_0^{\theta} e^{-x^2/2\sigma^2}\, dx\right)^{-1}.$$

By using formulas

$$t = T\tilde{t}, \quad x = L\tilde{x}, \quad u_k = L\tilde{u}_k, \quad T = \frac{2\pi}{\omega},$$

$$E_k = \tilde{E}_k \cdot 10^{11}\frac{\mathrm{g}}{\mathrm{s^2mm}}, \quad G_{12} = \tilde{G}_{12} \cdot 10^{11}\frac{\mathrm{g}}{\mathrm{s^2mm}}$$

we get the dimensionless form of Eqs. (3.1)

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial t^2} = \xi\Big(A_{1111}\dfrac{\partial^2 u_1}{\partial x_1^2} + A_{1212}\dfrac{\partial^2 u_1}{\partial x_2^2} + \big(A_{2211} + A_{1212}\big)\dfrac{\partial^2 u_2}{\partial x_1 \partial x_2}\Big), \\ \dfrac{\partial^2 u_2}{\partial t^2} = \xi\Big(\big(A_{2211} + A_{1212}\big)\dfrac{\partial u_1}{\partial x_1 \partial x_2} + A_{1212}\dfrac{\partial^2 u_2}{\partial x_1^2} + A_{2222}\dfrac{\partial^2 u_2}{\partial x_2^2}\Big), \end{cases} \tag{3.4}$$

with $\xi = \dfrac{T^2}{\rho L^2} \cdot 10^{14}$. Here the constants $A_{1111}, A_{2222}, A_{1212}$, and A_{2211} are expressed by formulas (3.3) with E_1, E_2, and G_{12} replaced by $\tilde{E}_1, \tilde{E}_2$, and $\tilde{G}_{12}$, respectively. For simplicity the tilde is omitted in Eqs. (3.4). The formulation of the waves interaction with a free boundary follows from (3.1)–(3.4) by setting $d = a$.

4. Numerical Results

We first solve the one-dimensional model. For solving of the problem (2.2) the explicit difference scheme was applied (see [4], pp. 332–355 and p. 161) and the following constants were used:

$$a = 1.5, \quad q = 20, \quad \omega = 300, \quad T = 2\pi/\omega, \quad b_1 = 20, \quad b_2 = 5 \ \text{ and } \ b_2 = 30.$$

Numerical results are exhibited in Figs. 4 and 5. In Fig. 4 ($b_1 = 20,\ b_2 = 5$) waves 1 and 4 exhibit the original waves, waves 2, 3 and 5 are reflected waves and function $u(t, x)$ is nonnegative. In Fig. 5 ($b_1 = 20,\ b_2 = 30$) waves 1 and

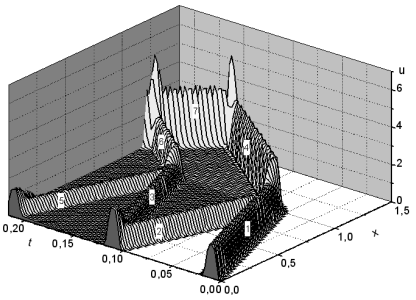

Figure 4. The graph of u for problem (2.2) with $b_1 = 20, b_2 = 5$.

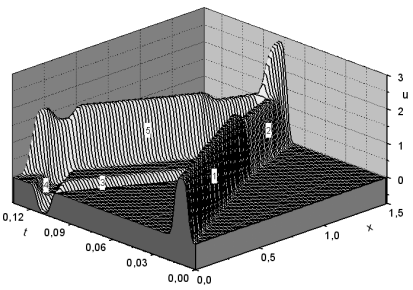

Figure 5. The graph of u for problem (2.2) with $b_1 = 20, b_2 = 30$.

2 are original waves while 3, 4 and 5 are reflected waves. For waves 3 and 4 function $u(x,t)$ is nonpositive.

To get the numerical solution to problem (3.4), (3.2) we use the finite-difference scheme [4]. Set

$$t_k = k\tau, \quad 0 \le k \le M, \quad \tau = T/M,$$
$$x_{1i} = -a + ih_1, \quad 0 \le i \le N_1, \quad h_1 = (d+a)/N_1, \quad N_1^t = 2a/h_1,$$
$$x_{2j} = -b + jh_2, \quad 0 \le j \le N_2, \quad h_2 = 2b/N_2.$$

Assume that $\bar{u}_{ij}^k = u_1(t_k, x_{1i}, x_{2j})$ and $\bar{\bar{u}}_{ij}^k = u_2(t_k, x_{1i}, x_{2j})$ and let

$$A_1 = \xi A_{1111}, \quad A_2 = \xi A_{1212}, \quad A_3 = \xi(A_{2211} + A_{1212}), \quad A_4 = \xi A_{2222}.$$

The explicit difference scheme is used to approximate the differential problem

$$\bar{u}_{ij}^{k+1} = b_{11}(\bar{u}_{i+1,j}^k - \bar{u}_{ij}^k) - b_{12}(\bar{u}_{i,j}^k - \bar{u}_{i-1,j}^k) + b_{13}(\bar{u}_{i,j-1}^k - 2\bar{u}_{ij}^k + \bar{u}_{i-1,j}^k)$$
$$+ b_{14}(\bar{\bar{u}}_{i+1,j+1}^k - \bar{\bar{u}}_{i+1,j-1}^k) - b_{15}(\bar{\bar{u}}_{i-1,j+1}^k - \bar{\bar{u}}_{i-1,j-1}^k) + 2\bar{u}_{ij}^k - \bar{u}_{ij}^{k-1},$$

$$\bar{\bar{u}}_{ij}^{k+1} = b_{14}(\bar{u}_{i+1,j+1}^k - \bar{u}_{i+1,j-1}^k) - b_{15}(\bar{u}_{i-1,j+1}^k - \bar{u}_{i-1,j-1}^k) + b_{21}(\bar{\bar{u}}_{i+1,j}^k - \bar{\bar{u}}_{i,j}^k)$$
$$- b_{22}(\bar{\bar{u}}_{i+1,j}^k - \bar{\bar{u}}_{i+1,j}) + b_{23}(\bar{\bar{u}}_{i,j-1}^k - 2\bar{\bar{u}}_{i,j}^k + \bar{\bar{u}}_{i,j+1}^k) + 2\bar{\bar{u}}_{i,j}^k - \bar{\bar{u}}_{i,j}^{k-1}.$$

The constants A_i $(i = 1, \ldots, 4)$ in Ω_1 and Ω_2 are different. Let A_i^- and A_i^+ be the A_i values in Ω_1 and Ω_2, respectively. To take into account the continuity conditions at the line $x_1 = a$, $x_2 \in (-b, b)$ we choose the coefficients b_{ij} as follows:

$$b_{11} = b_{12} = A_1^- \frac{\tau^2}{h_1^2}, \quad b_{13} = A_2^- \frac{\tau^2}{h_2^2}, \quad b_{14} = b_{15} = A_3^- \frac{\tau^2}{4h_1h_2},$$

$$b_{21} = b_{22} = A_2^- \frac{\tau^2}{h_1^2}, \quad b_{23} = A_4^- \frac{\tau^2}{h_1^2}, \quad \text{if} \quad i = 1, 2, \ldots, N_1^t,$$

$$b_{11} = \frac{A_1^+ + A_1^+}{2} \frac{\tau^2}{h_1^2}, \quad b_{12} = A_1^- \frac{\tau^2}{h_1^2}, \quad b_{13} = A_2^- \frac{\tau^2}{h_2^2}, \quad b_{14} = \frac{A_3^- + A_3^+}{2} \frac{\tau^2}{4h_1h_2},$$

$$b_{15} = A_3^- \frac{\tau^2}{4h_1h_2}, \quad b_{21} = \frac{A_2^- + A_2^+}{2}\frac{\tau^2}{h_1^2}, \quad b_{22} = A_4^- \frac{\tau^2}{h_2^2}, \quad b_{23} = A_4^- \frac{\tau^2}{h_2^2}, \quad \text{if } i = N_1^t,$$

$$b_{11} = A_1^+ \frac{\tau^2}{h_1^2}, \quad b_{12} = \frac{A_1^- + A_1^+}{2}\frac{\tau^2}{h_1^2}, \quad b_{13} = A_2^+ \frac{\tau^2}{h_2^2}, \quad b_{14} = A_3^+ \frac{\tau^2}{4h_1h_2},$$

$$b_{15} = \frac{A_3^- \tau^2}{4h_1h_2}, \quad b_{21} = A_2^+ \frac{\tau^2}{h_1^2}, \quad b_{22} = \frac{A_2^- + A_2^+}{2}\frac{\tau^2}{h_1^2}, \quad b_{23} = A_4^+ \frac{\tau^2}{h_2^2} \quad \text{if } i = N_1^t + 1,$$

$$b_{11} = b_{12} = A_1^+ \frac{\tau^2}{h_1^2}, \quad b_{13} = A_2^+ \frac{\tau^2}{h_1^2}, \quad b_{14} = b_{15} = A_3^+ \frac{\tau^2}{4h_1h_2},$$

$$b_{21} = b_{22} = A_2^+ \frac{\tau^2}{h_1^2}, \quad b_{23} = A_4^+ \frac{\tau^2}{h_2^2}, \quad \text{if } \ i = N_1^t + 2, \ldots, N_1 - 1.$$

Discretizing the boundary conditions we get

$$\begin{aligned} A_{1111}\frac{\bar{u}_{1j}^k - \bar{u}_{0j}^k}{h_1} + A_{2211}\frac{\bar{\bar{u}}_{0,j+1}^k - \bar{\bar{u}}_{0,j-1}^k}{2h_2} &= \mu_j^k \cos\alpha, \\ A_{1212}\frac{\bar{\bar{u}}_{0,j+1}^k - \bar{\bar{u}}_{0,j-1}^k}{2h_2} + \frac{\bar{u}_{1,j}^k - \bar{u}_{0,j}^k}{h_1} &= \mu_j^k \sin\alpha \end{aligned} \tag{4.1}$$

where

$$\mu_j^k = \begin{cases} B(t_k, x_{2j}), & \text{if } t_k \le 1, \text{ and } -b\varepsilon \le x_2 \le b\varepsilon, \\ 0 & \text{otherwise;} \end{cases}$$

$$A_{1111}\frac{\bar{u}_{N_1,j} - \bar{u}_{N_1-1,j}}{h_1} + A_{2211}\frac{\bar{\bar{u}}_{N_1,j+1} - \bar{\bar{u}}_{N_1,j-1}}{2h_2} = 0, \tag{4.2}$$

$$\frac{\bar{u}_{N_1,j+1} - \bar{u}_{N_1,j-1}}{2h_2} + \frac{\bar{\bar{u}}_{N_1,j} - \bar{\bar{u}}_{N_1-1,j-1}}{h_1} = 0, \quad \text{if } \ j = 1, 2, \ldots, N_2 - 1,$$

$$A_{2211}\frac{\bar{u}_{i+1,0} - \bar{u}_{i-1,0}}{2h_1} + A_{2222}\frac{\bar{\bar{u}}_{i,1} - \bar{\bar{u}}_{i,0}}{h_2} = 0, \tag{4.3}$$

$$\frac{\bar{u}_{i,1} - \bar{u}_{i,0}}{h_2} + \frac{\bar{\bar{u}}_{i+1,0} - \bar{\bar{u}}_{i-1,0}}{2h_1} = 0,$$

$$A_{2211}\frac{\bar{u}_{i+1,N_2} - \bar{u}_{i-1,N_2}}{2h_1} + A_{2222}\frac{\bar{\bar{u}}_{i,N_2} - \bar{\bar{u}}_{i,N_2-1}}{h_2} = 0, \tag{4.4}$$

$$\frac{\bar{u}_{i,N_2} - \bar{u}_{i,N_2-1}}{h_2} + \frac{\bar{\bar{u}}_{i+1,N_2} - \bar{\bar{u}}_{i-1,N_2}}{2h_1} = 0, \quad \text{if } \ i = 1, 2, \ldots, N_1 - 1.$$

To determine

$$\bar{u}_{0,0}^{k+1}, \ \bar{\bar{u}}_{0,0}^{k+1}, \ \bar{u}_{0,N_2}^{k+1}, \ \bar{\bar{u}}_{0,N_2}^{k+1}, \ \bar{u}_{N_1,0}^{k+1}, \ \bar{\bar{u}}_{N_1,0}^{k+1}, \ \bar{u}_{N_1,N_2}^{k+1}, \ \bar{\bar{u}}_{N_1,N_2}^{k+1}$$

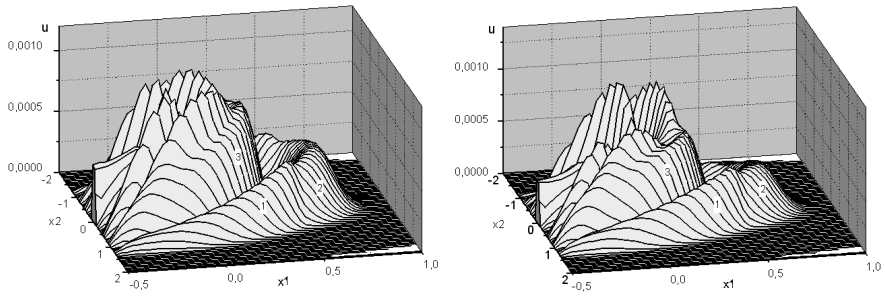

Figure 6. The graph of $|u|$. $\xi_1 = 0.2$, $\xi_2 = 0.4$, $t = 2$. (a) $\alpha = 0$; (b) $\alpha = \pi/6$.

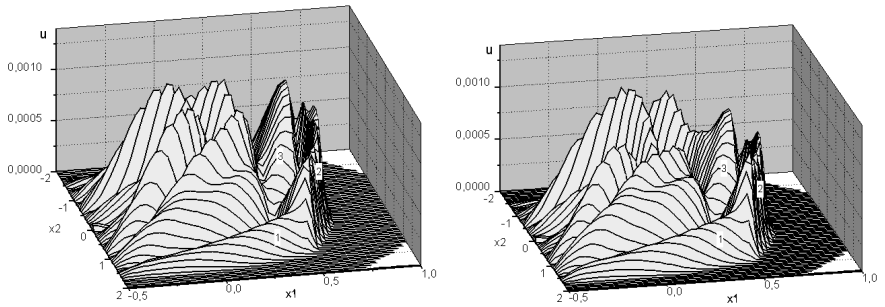

Figure 7. The graph of $|u|$. $\xi_1 = 0.3$, $\xi_2 = 0.032$, $t = 2$. (a) $\alpha = 0$; (b) $\alpha = \pi/6$.

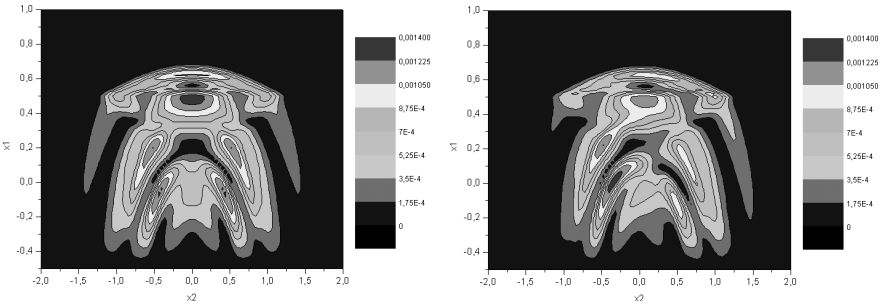

Figure 8. Isolines of $|u|$. $\xi_1 = 0.3$, $\xi_2 = 0.032$, $t = 2$. (a) $\alpha = 0$; (b) $\alpha = \pi/6$.

we use values from kth and $(k+1)$th time layers that are already determined:

$$\bar{u}_{0,0}^{k+1} = \frac{\bar{u}_{0,0}^{k} + \bar{u}_{0,1}^{k} + \bar{u}_{1,0}^{k} + \bar{u}_{1,1}^{k+1}}{4}, \quad \bar{\bar{u}}_{0,N_2}^{k+1} = \frac{\bar{\bar{u}}_{0,N_2}^{k} + \bar{\bar{u}}_{0,N_2-1}^{k} + \bar{\bar{u}}_{1,N_2}^{k} + \bar{\bar{u}}_{1,N_2-1}^{k+1}}{4}.$$

The discretized initial conditions are the following

$$\bar{u}_{ij}^0 = 0, \quad \bar{\bar{u}}_{ij}^0 = 0, \quad \bar{u}_{ij}^1 = \bar{u}_{ij}^0, \quad \bar{\bar{u}}_{ij}^1 = \bar{\bar{u}}_{ij}^0, \tag{4.5}$$

$$i = 0, 1, \ldots, N_1, \ j = 0, 1, \ldots, N_2.$$

This allows us to solve four systems of linear equations (4.1), (4.2), (4.3) and (4.4) separately and so essentially reduce CPU time. The matrix of each such system consists of four differently located blocks. Two blocks are diagonal matrixes and the other two ones are of two diagonal type.

The local approximation error for inner points of $\Omega_1 \cup \Omega_2$ is $O(\tau^2 + h_1^2 + h_2^2)$. The discrete boundary conditions are given with the accuracy $O(h_1 + h_2^2)$ at the lines $x_1 = -a$ and $x_1 = d$ and $O(h_1^2 + h_2)$ at $x_2 = \pm b$, respectively. At the line $x_1 = a$ we have the local approximation of the second order with respect to t and x_2. The difference sheme is stable if the grid parameters h_1, h_2, and τ satisfy the inequality

$$\tau = \frac{\min(h_1, h_2)}{\sqrt{4 \max_{i=1,\ldots,4} \left(A_i^-, A_i^+\right)}}.$$

We performed several experiments with different values of τ, h_1, and h_2. The results presented in this section are computed with

$$a = 0.5, \ d = 1, \ b = 2, \ h_1 = 0.01, \ h_1 = 0.02, \ \tau = 0.005.$$

The numerical experiments for different values of τ, h_1, and h_2 show that the numerical solution converges to the solution of problem (3.1) and (3.2).

In the 2-dimensional case we used the following constants:

$$T = 10^{-5}\,\mathrm{s}, \quad L = 200\,\mathrm{mm}, \quad \rho = 7.9 \cdot 10^{-3}\,\mathrm{g/mm}^3, \ \eta = 1000, \ \varepsilon = 0.1,$$
$$\tilde{E}_1 = 2.3, \quad \tilde{E}_2 = 1.9, \quad \tilde{G}_{12} = 0.8, \quad \xi_1 = 0.3, \quad \xi_2 = 0.032$$

and $\xi_1 = 0.2$, $\xi_2 = 0.4$. Numerical results of the waves diffraction problem are presented in Figs. 6–9. In Fig. 6(a) the graph of $|u|$ is presented for $\xi_1 = 0.2$, $\xi_2 = 0.4$, $\alpha = 0$, and $t = 2$. Fig. 6(b) represents the graph of $|u|$ for $\xi_1 = 0.2$, $\xi_2 = 0.4$, $\alpha = \pi/6$, and $t = 2$. In both 6(a) and 6(b) figures waves 1 and 2 illustrate longitudinal waves ($\operatorname{div} u \neq 0$) while waves 3 are transversal ($\operatorname{div} u = 0$). In Figs. 7(a) and 7(b) the graph of $|u|$ is represented for $t = 2$, $\xi_1 = 0.3$, $\xi_2 = 0.032$, $\alpha = 0$, and $\alpha = \pi/6$, respectively. Here waves 1 and 2 are longitudinal and waves 3 are transversal. Figures 6 and 7 show, that support of wave in domain Ω_2 decreases with ξ_2/ξ_1 decreasing. In Figs. 8(a) and 8(b) the isolines of $|u|$ are represented for data given in Figs. 7(a) and 7(b).

Figures 9(a)–(f) represent the dynamics of the waves interaction with the free boundary ($x_1 = d = a$) for $\alpha = 0, \xi = 0.032$, and $t = 4$, 4.25, 4.5, 4.75, 5 and 5.25, respectively. In Figs. 9(d)–(f) we can see also the interaction of waves moving forward and back.

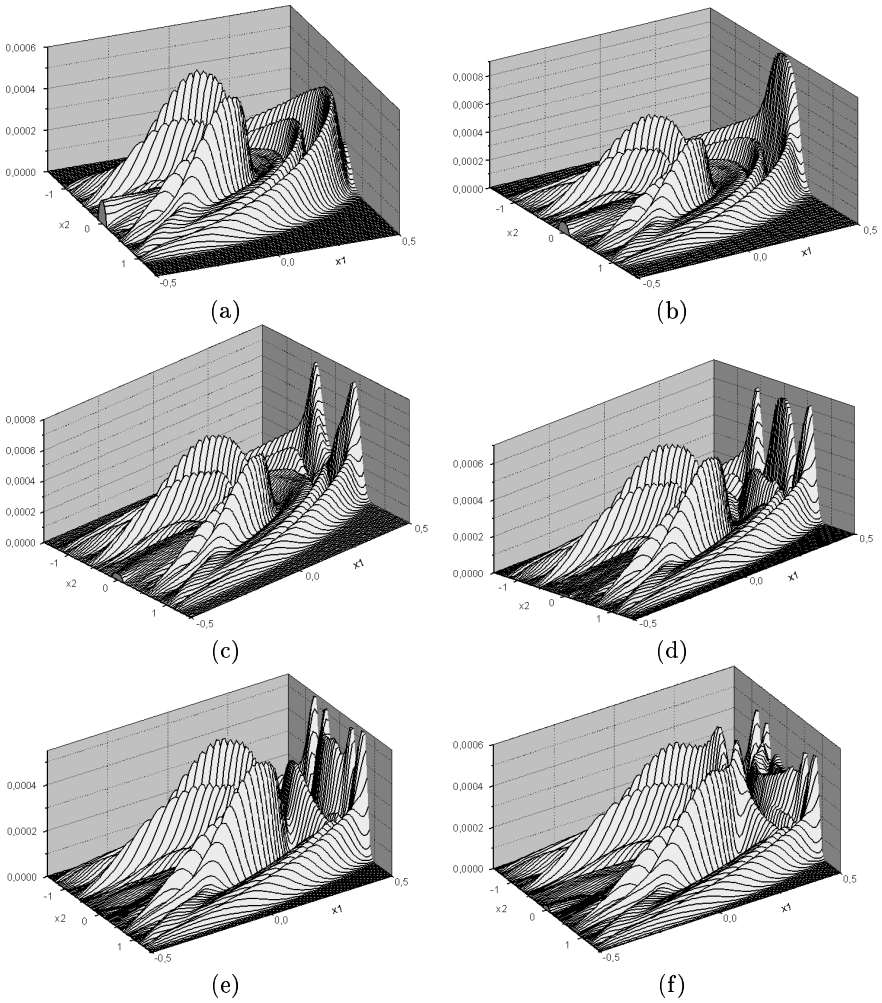

Figure 9. The graph of $|u|$. $\xi = 0.032$, (a) $t = 4$; (b) $t = 4.25$; (c) $t = 4.5$; (d) $t = 4.75$; (e) $t = 5$; (f) $t = 5.25$.

5. Concluding Remarks

1. The diffraction and waves interaction with a free boundary problems are examined numerically and analytically in a material structure composed of random oriented orthotropic crystals. In the two-dimensional case only the numerical solution is examined.
2. The dynamics of the longitudinal and transversal waves for both problems are studied.
3. The waves increase (decrease) in the domains Ω_2 and $(1, d)$ with $\xi_2 < \xi_1$ and $b_2 < b_1$ ($\xi_2 > \xi_1$ and $b_2 > b_1$), respectively, is obtained. Because of the interference the waves increase at the free boundary in one-dimensional problem is observed.

4. In two-dimensional problem the waves interaction with the free boundary is more complicated. Since waves are not plane, then different parts of these waves reach the free boundary at the different moments. Some parts of a wave move forward while the other ones move back. The interaction of reflected waves and the other ones moving forward complicates the analysis (see Figs. 9(d)–(f)).
5. In the symmetric case the wave dynamics on the line of symmetry is similar to that of the one-dimensional problem.

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