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ON THE POPULATION MODEL WITH A SINE FUNCTION

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Abstract. In the interval $[0,1]$ function $s_r(x) = r\sin\pi x$ behaves similar to logistic function $h_\mu(x) = \mu x(1-x)$. We prove that for every $r > \frac{\sqrt{\pi^2+1}}{\pi}$ there exists subset $\Lambda \subset [0,1]$ such that $s_r : \Lambda \to \Lambda$ is a chaotic function. Since the logistic function is chaotic in another subset of $[0,1]$ but both functions have similar graphs in $[0,1]$ we conclude that it can lead to errors in practice.

Key words: logistic function, sine function, chaotic function

1. Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time. If a certain population has discrete generation, the size of the $(n+1)$-th generation $x(n+1)$ is a function of the nth generation $x(n)$. This relation expresses itself in the difference equation

$$x(n+1) = f(x(n)).$$

We start in the point x_0 and use the following notation

$$f^2(x_0) = f(f(x_0)), \quad f^3(x_0) = f(f(f(x_0))).$$

Letting $x(n) = f^n(x_0)$, we have

$$x(n+1) = f^{n+1}(x_0) = f(f^n(x_0)) = f(x(n)).$$

The exponential models have only a limited predictive power in population problems since as time passes the predicted population becomes so large that it is no longer realistic. For most biological species it is valid that the population increases until it reaches a certain upper limit. Then, due to the limitations of available resources, the creatures will become testy and engage in competition for those limited resources. In 1845 Pierre-Francois Verhulst had offered for investigation of population the following mathematical model

$$x(n+1) = \mu x(n)(1 - x(n)),$$

where $x(n)$ is the size of a population at time n and μ is the rate of growth of the population. This equation is the simplest nonlinear first-order difference equation but it describes a complicated dynamics. The quadratic function $h\mu(x) = \mu x(1-x)$, $x \in [0,1]$, is called also ***logistic function.*** The logistic function is widely studied (see, for example, [3, 4, 5, 6]).

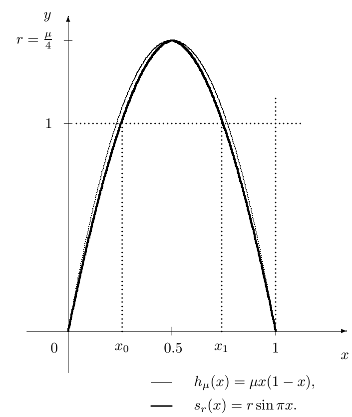

Figure 1. The logistic and sin functions.

The function

$$s_r(x) = r \sin \pi x$$

also has a similar behaviour in the interval $[0,1]$, here $r > 0$ is a parameter (see Fig. 1). If we fix x_0 and consider the orbit

$$\{x_0, \ \ s_r(x_0), \ \ s_r(s_r(x_0)), ..., \}$$

then similar to logistic function the dynamics of $s_r(x)$ is very complicated. There exist more functions with similar behaviour in [0,1], but we consider *sine* function, because it is very popular to approximate real functions with trigonometric series, especially by using Fourier series. Let $D \subset \mathbf{R}$.

DEFINITION 1. (R.Devaney, [2]) The function $f : D \to D$ is *chaotic* if

a) the periodic points of f are dense in D,

b) f is topologically transitive in D,

c) f exhibits sensitive dependence on initial conditions in D.

We prove that for every $r > \frac{\sqrt{\pi^2+1}}{\pi}$ there exists a subset $\Lambda \subset [0,1]$ such that $s_r : \Lambda \to \Lambda$ is chaotic function by Devaney's definition. The chaotic function preserves a certain amount of regularity and mixes the domain well. Even very small changes in the initial position may result in dramatically different results in values of the iterated function.

2. The Dynamics of Sine Function

In this section we prove that for every $r > \frac{\sqrt{\pi^2+1}}{\pi}$ there exists a subset $\Lambda \in [0,1]$ such that $s_r : \Lambda \to \Lambda$ is a chaotic function. At first, we clear up conceptions of the Devaney definition of chaotic functions (see [3]). Let $D \subset \mathbf{R}$.

DEFINITION 2. Let A be a subset of $B \subset \mathbf{R}$. Then A is *dense* in B if for each point $x \in B$ and each $\varepsilon > 0$ there exists $y \in A$ such that $|x - y| < \varepsilon$.

DEFINITION 3. The function $f : D \to D$ is *topologically transitive* on D if for any two points x and y in D and any $\varepsilon > 0$, there exists z in D such that $|z - x| < \varepsilon$ and $|f^n(z) - y| < \varepsilon$ for some n.

DEFINITION 4. The function $f : D \to D$ *exhibits sensitive dependence on initial conditions* if there exists a δ such that for any x in D and any $\varepsilon > 0$, there is a y in D and a natural number n such that $|x - y| < \varepsilon$ and $|f^n(x) - f^n(y)| > \delta$.

The result by Banks, Brooks, Cairns, Davis and Stacey [1] demonstrates that when the domain of definition of a continuous function is infinite, then the density of periodic points and topological transitivity imply sensitive dependence on initial conditions.

Theorem 1. ([1]) *Let D be an infinite subset of the real numbers and $f : D \to D$ be continuous. If f is topologically transitive on D and the periodic points of f are dense in D, then f is chaotic on D.*

In our case $s_r(x) = r \sin \pi x$ is continuous function in segment $[0,1]$. If the function f is differentiable at each point of $D \subset \mathbf{R}$ and both f and f' are continuous then f is said to be *continuously differentiable* or to be a C^1 function. For this class of functions the following result is valid.

Theorem 2. ([5], p.38-39, Theorem 5.2 and 5.3) *Let $f : \mathbf{R} \to \mathbf{R}$ be a C^1 function and $I_1, \dots, I_p$ be p disjoint closed bounded intervals with $p \geq 2$. Let $J = \cup_{j=1}^{p} I_j$. Assume that $f(I_j) \supset J$ for $1 \leq j \leq p$. Also assume that there is a constant $\lambda > 1$ such that $|f'(x)| \geq \lambda$ for $x \in J \cap f^{-1}(J)$. Let $\Lambda = \cap_{k=0}^{\infty} f^{-k}(J)$. Then the following statements are valid: a) Λ is a Cantor set, b) the set of periodic points of f is dense in Λ, c)f is topologically transitive on Λ.*

We consider the set $\Lambda_1 = [0, \frac{1}{\pi} \arcsin \frac{1}{r}] \cup [1 - \frac{1}{\pi} \arcsin \frac{1}{r}, 1]$ (or $\Lambda_1 = \{x | s_r(x) \text{ is in } [0,1]\}$, $r > 1$). Now we prove the following proposition.

Proposition 1. *If* $s_r(x) = r \sin \pi x$ *and* $r > \dfrac{\sqrt{\pi^2+1}}{\pi} \approx 1.05$, *then*

$$\inf\{|s_r'(x)|| x \in \Lambda_1\} > 1.$$

Proof. The derivative of s_r is given by $s_r'(x) = r\pi \cos \pi x$. Also the estimate $s_r''(x) = -r\pi^2 \sin \pi x < 0$ is valid for all $x \in]0,1[$. So the smallest value of $|s_r'(x)|$ on Λ_1 occurs where $s_r(x) = 1$. Solving equation

$$1 = s_r(x) = r \sin \pi x$$

we get $x_0 = \dfrac{1}{\pi} \arcsin \dfrac{1}{r}$ and $x_1 = 1 - \dfrac{1}{\pi} \arcsin \dfrac{1}{r}$ (see Fig.1). For these points we have

$$s_r'(x_0) = r\pi \cos\left(\pi \cdot \frac{1}{\pi} \arcsin \frac{1}{r}\right) = \pi\sqrt{r^2-1},$$

$$s_r'(x_1) = r\pi \cos\left(\pi \cdot (1 - \frac{1}{\pi} \arcsin \frac{1}{r})\right) = -\pi\sqrt{r^2-1}.$$

We need to satisfy the inequality $| s_r'(x_{0,1}) | = \pi\sqrt{r^2-1} > 1$. From it we find $r > \dfrac{\sqrt{\pi^2+1}}{\pi}$. Therefore if $r > \dfrac{\sqrt{\pi^2+1}}{\pi}$ then

$$\inf\{|s_r'(x)| \mid x \in \Lambda_1\} > 1.$$

■

Theorem 3. *If* $r > \dfrac{\sqrt{\pi^2+1}}{\pi} > 1$, *then there exist a subset* $\Lambda \subset [0,1]$ *such that* $s_r : \Lambda \to \Lambda$ *is a chaotic function.*

Proof. The function $s_r(x) = r \sin \pi x$ is a C^1 function. Since

$$x_0 = \frac{1}{\pi} \arcsin \frac{1}{r} < \frac{1}{2} < x_1 = 1 - \frac{1}{\pi} \arcsin \frac{1}{r}$$

(see Fig.1), then

$$I_1 = [0, \frac{1}{\pi} \arcsin \frac{1}{r}] \text{ and } I_2 = [1 - \frac{1}{\pi} \arcsin \frac{1}{r}, 1]$$

are two disjoint closed bounded intervals. Besides $s_r(I_j) \supset I_1 \cup I_2$, $j = 1, 2$. By Proposition 1 there is a $\lambda_r = \inf\{|s_r'(x)| \mid x \in \Lambda_1\} > 1$ such that for $x \in (I_1 \cup I_2) \cap s_r^{-1}(I_1 \cup I_2)$ we have $|s_r'(x)| \geq \lambda_r$. Then by Theorem 2

$$\Lambda = \cap_{k=0}^{\infty} s_r^{-k}(I_1 \cup I_2)$$

is a Cantor set, that is, it is an infinite subset of the real numbers, and $s_r : \Lambda \to \Lambda$ is continuous. By Theorem 2 the set of periodic points of s_r are dense in Λ and s_r is topologically transitive on Λ. From Theorem 1 it follows that s_r is chaotic on Λ. ■

3. Comparison of Sine Function with Logistic Function

We make three remarks.

1. Suppose we have 100 points of graph of function $s_r(x) = r \sin \pi x$, $r > \frac{\sqrt{\pi^2+1}}{\pi}$, and we do not know that exactly this function is considered. What is a possibility that if we make approximation we choose s_r? Graphics of s_r and h_μ are similar (see Fig. 1 in case $r = \frac{\mu}{4}$). Since quadratic function is more simple, therefore it is high possibility that we make approximation with quadratic map. If this quadratic map is $h_\mu = \mu x(1-x)$, $\mu > 4$, then we know that there exists a subset $\Lambda_h \subset [0,1]$ such that $h_\mu : \Lambda_h \to \Lambda_h$ is chaotic ([3, 5]). Also there exists a subset $\Lambda_s \subset [0,1]$ such that $s_r : \Lambda_s \to \Lambda_s$ is chaotic. But $\Lambda_h \neq \Lambda_s$ (for example, if $r = \dfrac{\mu}{4}$ then
$$x_0 = \frac{1}{\pi} \arcsin \frac{1}{r}, \quad x_1 = 1 - \frac{1}{\pi} \arcsin \frac{1}{r}$$
belong to Λ_s but not to Λ_h).

2. The function $s_r(x) = r \sin \pi x$, $x \in [0,1]$, has the following properties:
   (1) it is smooth function and $s_r : [0,1] \to \mathbf{R}$,
   (2) it has maximum point $x_m = \dfrac{1}{2}$ and $s_r''\big(\dfrac{1}{2}\big) = -r\pi^2 \neq 0$,
   (3) it is monotone in segments $[0, x_m[$ and $]x_m, 1]$,
   (4) it has negative the Schwarz derivative, i.e., $\forall x \in [0,1] \setminus \{x_m\}$
$$S_{s_r}(x) = \frac{s_r'''(x)}{s_r'(x)} - \frac{3}{2}\left(\frac{s_r''(x)}{s_r'(x)}\right)^2 = -\pi^2\left(1 + \frac{3}{2}\left(\frac{\sin \pi x}{\cos \pi x}\right)^2\right) < 0.$$
   By [4] in this case there exists bifurcation diagram for s_r and it is similar to h_μ.

3. Let $h_\mu(x) = \mu x(1-x)$, $\mu > 4$, and let $\Lambda_h \subset [0,1]$ is a set in which h_μ is chaotic. It is known that if $x \notin \Lambda_h$ then the limit of $h_\mu^n(x)$ is equal to minus infinity as n goes to infinity ([3, 5, 6]). Let $s_r(x) = r \sin \pi x$, $r > \frac{\sqrt{\pi^2+1}}{\pi}$, and let $\Lambda_s \subset [0,1]$ is a set in which h_r is chaotic. Since $|r \sin \pi x| \leq 1$ then if $x \notin \Lambda_s$ then the limit of $s_r^n(x)$ is not infinity as n goes to infinity. This is a very interesting difference.

   For example, we consider point $x_m = \dfrac{1}{2}$. If $r \in \mathbf{N}$ then
$$s_r\big(\frac{1}{2}\big) = r > 1, \;\; s_r^2\big(\frac{1}{2}\big) = r \sin \pi r = 0, \;\; s_r^k\big(\frac{1}{2}\big) = 0, \;\; k \geq 2,$$

i.e., x_m is eventually fixed point and x_m returns to the interval $[0,1]$ under iteration of s_r. If $r = \frac{m}{2}$, $m \in \mathbf{N}$ and m is an odd number then
$$s_r\big(\frac{1}{2}\big) = \frac{m}{2}, \; s_r^2\big(\frac{1}{2}\big) = \frac{m}{2} \sin \frac{\pi m}{2} = \begin{cases} \dfrac{m}{2}, & m = 1 + 4l, l \in \mathbf{N}, \\ -\dfrac{m}{2}, & m = 3 + 4l, l \in \mathbf{N}, \end{cases} \qquad s_r^3\big(\frac{1}{2}\big) = \frac{m}{2}.$$

We see that if $r = \frac{1+4l}{2}$, $l \in \mathbf{N}$, then $x_m = \frac{1}{2}$ is eventually a fixed point but it does not belong to $[0,1]$. If $r = \frac{3+4l}{2}$, $l \in \mathbf{N}$, then $x_m = \frac{1}{2}$ is eventually a periodic point with period 2 and it also leaves the interval $[0,1]$ under iteration of s_r.

Generally, if $r > 2$ then there exist two points $x_1, x_2 \in [0,1]$ such that

$$s_r(x_1) = s_r(x_2) = 2, \quad s_r^2(x_1) = s_r^2(x_2) = r \sin r\pi = 0,$$

i.e., x_1 and x_2 are eventually fixed points and they return back to the interval $[0,1]$. We also remark that if $r > 2$ then for every $y \in]1,2]$ there exist x_3 and x_4 in $[0,1]$ such that

$$s_r(x_3) = s_r(x_4) = y, \quad s_r^2(x_3) = s_r^2(x_4) \in [-r, 0].$$

Since $s_r([-1,0]) = [-r,0]$ and $s_r([-2,-1]) = [0,r]$ then there exist points such that they leave the interval $[0,1]$ in the first iteration and return back to the interval $[0,1]$ under iteration of s_r.

4. Conclusion

We conclude that functions with similar graphs have different "chaotic" properties. If we make approximation of real data with one chaotic function but in reality this data was generated by another chaotic function then it is possible that the errors of our forecast are higher than we expected. In the case of chaotic functions the long-term forecast is not possible and we can not define precisely exact functions thus the choice of models with chaotic functions is unpredictable.

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