Mathematical Modelling and Analysis
Volume 22 Number 3, May 2017, 300–310
https://doi.org/10.3846/13926292.2017.1307790

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
ISSN: 1392-6292
eISSN: 1648-3510

Two-Scale Estimates for Special Finite Discrete Operators*

Alexander V. Vasilyev[a,b] **and Vladimir B. Vasilyev**[a,b]

[a] *Belgorod National Research University*
Studencheskaya 14/1, 308007 Belgorod, Russia
[b] *Lipetsk State Technical University*
E-mail(*corresp.*): vladimir.b.vasilyev@gmail.com
E-mail: alexvassel@gmail.com

Received October 9, 2016; revised March 4, 2017; published online May 15, 2017

Abstract. We consider a certain finite discrete approximation for multidimensional Calderon–Zygmund integral operator and give a comparison between solutions of corresponding equations in some spaces of discrete functions.

Keywords: finite discrete operator, Calderon–Zygmund operator, error estimate.

AMS Subject Classification: 42B20; 65R20.

1 Introduction

We consider the following Calderon–Zygmund singular integral operator [10]

$$(Ku)(x) \equiv v.p. \int_{\mathbb{R}^m} K(x-y)u(y)dy, \quad x \in \mathbb{R}^m, \tag{1.1}$$

in the Lebesgue space $L_2(\mathbb{R}^m)$ and its discrete analogue of the following type

$$(K_d u_d)(\tilde{x}) \equiv \sum_{\tilde{y} \in h\mathbb{Z}^m} K_d(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m, \quad \tilde{x} \in h\mathbb{Z}^m, \tag{1.2}$$

in the space $L_2(h\mathbb{Z}^m) \equiv l^2$ of functions u_d of a discrete variable $\tilde{x} \in h\mathbb{Z}^m$.

We recall [10] that that a symbol $\sigma(\xi)$ of the operator K is the Fourier transform of its kernel in principal value sense

$$\sigma(\xi) = \lim_{\varepsilon \to 0, N \to \ +\infty} \int\limits_{\varepsilon < |x| < N} K(x)e^{ix\cdot\xi}dx,$$

* This work was supported by RFBR and Lipetsk regional government of Russia, grant No. 14-41-03595-r-center-a

and such symbol is called an elliptic symbol if

$$\inf_{\xi} |\sigma(\xi)| > 0.$$

Our main goal is the following. Starting from the operator (1.1) we introduce the discrete operator (1.2) acting in infinite dimensional space so that it preserves basic properties of the operator (1.1) related to an ellipticity and invertibility [18, 19, 20, 21, 22]. Further to obtain computational algorithms we would like to construct finite dimensional analogue of the operator (1.2) preserving same properties, and to obtain results on comparison of these operators and solutions of corresponding equations.

1.1 Some previous approaches and studies

In books [2, 3, 5, 6, 14, 15, 17] authors present studies on convolution equations, one-dimensional singular integral equations and multidimensional weakly singular integral equations related to approximate solution of these equations.

As usual there are the following questions in these studies: 1) to find an approximate equation desirable in a finite-dimensional (N-dimensional) space so that under enough large N this approximate equation will be uniquely solvable in an appropriate space; 2) to obtain an error estimate between exact solution of an initial equation and exact solution of approximate equation in dependence on N. As far as we know these questions were investigated fully for one-dimensional singular integral equations on smooth curves L in a complex plane $\mathbb{C}$ [8, 15]

$$a(t)u(t) + \frac{b(t)}{\pi i} v.p. \int_L \frac{u(\tau)}{t-\tau} d\tau = v(t), \quad t \in L,$$

convolution equations with integrable kernel $K(x)$ on a straight line

$$a(x)u(x) + \int_{-\infty}^{+\infty} K(x-y)u(y)dy = v(x), \quad x \in \mathbb{R},$$

and some their multidimensional analogues [2, 3, 5] and for multidimensional integral equations with a weak singularity

$$a(x)u(x) + \int_D K(x,y)u(y)dy = v(x), \quad x \in D \subset \mathbb{R}^m,$$

where D is a domain with a smooth boundary ∂D and the kernel $K(x,y)$ satisfies the estimate

$$|K(x,y)| \leq \frac{c}{|x-y|^{m-\alpha}}, \quad 0 < \alpha \leq 1;$$

last equations are generated by compact integral operators in appropriate spaces [17].

An algebraic approach based on a local principle for studying certain finite approximations for integral operators was used in many papers, and its development for last years is presented in books for example [2, 3, 6]. But this method of C^*-algebras permits to prove a solvability of approximating equation but it can't help for obtaining an error estimate for solutions of these equations. Moreover concrete applications of the method are related as a rule to one-dimensional singular integral operators and equations.

For an error estimate there are some computational results [9,12,13,16], and a lot of results are related to special equations for applied problems (see for example [7]). Multidimensional case is more complicated because there is no such advanced theory similar classical Riemann boundary value problem [4,11].

We would like to note that in general there are no valuable results for approximate solution of multidimensional singular integral equations with Calderon – Zygmund operators although such equations in distinct form arise in many problems of partial differential equations and mathematical physics [10]. In our opinion a direct digitization is more convenient for computer calculations than other methods, and we try to use and justify this approach.

2 Discrete spaces and transformations

2.1 Definitions and notations

We accept the following conventions and notations. The kernel $K(x)$ is a differentiable function on the unit sphere in $\mathbb{R}^m, K(0) \equiv 0$, and K_d is a restriction of the kernel K on lattice points $h\mathbb{Z}^m$. Let h be a size of a "spatial quant", N be a size of our "Universe". These parameters will be tend to zero and infinity respectively. For the operator (1.1) we'll use the following reduction. First we replace the operator (1.1) by the discrete operator (1.2), and second we approximate this series by a special finite sum

$$\sum_{\tilde{y}\in h\mathbb{Z}^m\cap Q_N} K_{d,N}(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m, \quad \tilde{x}\in h\mathbb{Z}^m\cap Q_N. \tag{2.1}$$

For $K_{d,N}$ in the formula (2.1) we suggest a following construction. If K_d is a restriction of the continual kernel K on lattice points $h\mathbb{Z}^m$ then we take a restriction of the K_d on points $h\mathbb{Z}^m \cap Q_N$, where

$$Q_N = \{x\in\mathbb{R}^m : x=(x_1,\cdots,x_m), \max_{1\le k\le m}|x_k|\le N\}$$

and denote by $K_{d,N}$ its periodic continuation on the whole $h\mathbb{Z}^m$.

We would like to justify a following sequence of transformations, "continual" operator (1.1) $\longrightarrow$ "infinite discrete" operator (1.2) $\longrightarrow$ "finite discrete" operator (2.1) with corresponding estimates on h and N. The comparison of (1.1) and (1.2) was given in papers' series [18,19,20,21,22], and here we consider a comparison between (1.2) and (2.1).

Let's denote by P_N the restriction operator $h\mathbb{Z}^m \longrightarrow h\mathbb{Z}^m\cap Q_N$, and the space $L_2(h\mathbb{Z}^m\cap Q_N)$ is denoted by l^2_N so that P_N is a projector $l^2\to l^2_N$.

Definition 1. Approximation rate of operators K_d and $K_{d,N}$ is called the following operator norm

$$||K_{d,N}P_N - P_NK_d||_{l^2\to l^2_N}.$$

It is non-trivial to obtain an estimate for the operator norm, but we'll give an estimate for an individual element assuming the existence of some its properties. More precisely we'll suppose that the element u_d is a restriction of the function u which has Hölder property in $\mathbb{R}^m$ and $u(x) = o(|x|^\gamma), |x| \to \infty$, with some $\gamma > 0$.

We will obtain a "weak estimate" for approximation rate but enough for our purposes. We assume additionally that a function u_d is a restriction on $h\mathbb{Z}^m$ of continuous function with certain estimates [20, 21]. Let's define the discrete space $C_h(\alpha, \beta)$ as a functional space of discrete variable $\tilde{x} \in h\mathbb{Z}^m$ with finite norm

$$||u_d||_{C_h(\alpha,\beta)} = ||u_d||_{C_h} + \sup_{\tilde{x},\tilde{y}\in h\mathbb{Z}^m} \frac{|\tilde{x}-\tilde{y}|^\alpha}{(\max\{1+|\tilde{x}|,\, 1+|\tilde{y}|\})^\beta}.$$

It means that the function $u_d \in C_h(\alpha, \beta)$ satisfies the following estimates

$$|u_d(\tilde{x}) - u_d(\tilde{y})| \le c\, \frac{|\tilde{x}-\tilde{y}|^\alpha}{(\max\{1+|\tilde{x}|,\, 1+|\tilde{y}|\})^\beta},$$
$$|u_d(\tilde{x})| \le \frac{c}{(1+|\tilde{x}|)^{\beta-\alpha}}, \quad \forall \tilde{x}, \tilde{y} \in h\mathbb{Z}^m,\ \alpha, \beta - \alpha > m,\ 0 < \alpha < 1.$$

Let us note that under required assumptions $C_h(\alpha, \beta) \subset L_2(h\mathbb{Z}^m)$, and these discrete space are discrete analogue of corresponding subspaces of continuous functions [1].

Theorem 1. *For operators K_d and $K_{d,N}$ the following estimate*

$$||K_{d,N}P_N u_d - P_N K_d u_d||_{l^2_N} \le C\, N^{m+2(\alpha-\beta)}$$

holds, where constant C doesn't depend on N, h.

Proof. Let us write

$$(P_N K_d - K_{d,N}P_N)u_d = P_N K_d P_N u_d - K_{d,N} P_N u_d + P_N K_d (I - P_N) u_d,$$

where I is an identity operator in $L_2(h\mathbb{Z}^m)$.

First two summands have annihilated, and we need to estimate only the last summand. We have

$$||P_N K_d (I - P_N) u_d|| \le C||(I - P_N) u_d||$$

because norms of operators K_d are uniformly bounded, and for the last norm taking into account above estimates we can write

$$||(I - P_N)u_d||^2 \le C \sum_{\tilde{x}\in h\mathbb{Z}^m\setminus Q_N} |u_d(\tilde{x})|^2 h^m \le C \sum_{\tilde{x}\in h\mathbb{Z}^m\setminus Q_N} \frac{h^m}{(1+|\tilde{x}|)^{2(\beta-\alpha)}} \le$$

and further

$$C \int_{\mathbb{R}^m\setminus Q_N} |x|^{2(\alpha-\beta)} dx.$$

The last integral using spherical coordinates gives the estimate $N^{m+2(\alpha-\beta)}$ which tends to 0 under $n \to \infty$ if $\beta > \alpha + m/2$. □

Remark 1. Similar theorem was obtained in [19, 20] for the space $C_h(\alpha,\beta)$.

This theorem plays a key role for obtaining an estimate for approximate solution of an multidimensional singular integral equation with the operator (1.1) and permits to use fast Fourier transform for evaluating a numerical solution. Some test calculations were given in [20].

2.2 Discrete Fourier transform

We define the discrete Fourier transform for a function u_d of a discrete variable $\tilde{x}\in h\mathbb{Z}^m$ as the series

$$\tilde{u}_d(\xi)=\sum_{\tilde{x}\in h\mathbb{Z}^m} e^{-ix\cdot\xi}u_d(\tilde{x})h^m,\quad \xi\in\hbar\mathbb{T}^m,$$

where $\hbar=h^{-1}/(2\pi)$.

This discrete Fourier transform has same properties like standard continual Fourier transform, particularly for a discrete convolution of two discrete functions u_d, v_d

$$(u_d*v_d)(\tilde{x})\equiv\sum_{\tilde{y}\in h\mathbb{Z}^m}u_d(\tilde{x}-\tilde{y})v_d(\tilde{y})h^m$$

we have the well known multiplication property

$$(F_d(u_d*v_d))(\xi)=(F_du_d)(\xi)\cdot(F_dv_d)(\xi).$$

If we apply this property to the operator K_d we obtain

$$(F_d(K_du_d))(\xi)=(F_dK_d)(\xi)\cdot(F_du_d)(\xi).$$

Let us denote $(F_dK_d)(\xi)\equiv\sigma_d(\xi)$ and give the following

Definition 2. The function $\sigma_d(\xi),\xi\in\hbar\mathbb{T}^m$, is called a periodic symbol of the operator K_d.

We will assume below that the symbol $\sigma_d(\xi)\in C(\hbar\mathbb{T}^m)$ therefore we have immediately the following

Property 1. The operator K_d is invertible in the space $L_2(h\mathbb{Z}^m)$ iff $\sigma_d(\xi)\neq 0,\forall\xi\in\hbar\mathbb{T}^m$.

Definition 3. A continuous periodic symbol is called an elliptic symbol if $\sigma_d(\xi)\neq 0,\forall\xi\in\hbar\mathbb{T}^m$.

So we see that an arbitrary elliptic periodic symbol $\sigma_d(\xi)$ corresponds to an invertible operator K_d in the space $L_2(h\mathbb{Z}^m)$.

Remark 2. It was proved earlier that operators (1.1) and (1.2) for cases $D=\mathbb{R}^m, D=\mathbb{R}^m_+$ are invertible or non-invertible in spaces $L_2(\mathbb{R}^m), L_2(\mathbb{R}^m_+)$ and $L_2(h\mathbb{Z}^m), L_2(h\mathbb{Z}^m_+)$ simultaneously [18, 22].

2.3 Finite discrete Fourier transform

DEFINITION 4. For a function $u_{d,N} \in l^2_N$ its finite discrete Fourier transform is defined by the formula

$$\tilde{u}_{d,N}(\tilde{\xi}) = \sum_{\tilde{x} \in h\mathbb{Z}^m \cap Q_N} e^{i\tilde{x}\cdot\tilde{\xi}} u_{d,N}(\tilde{x}) h^m, \quad \tilde{\xi} \in \hbar\mathbb{Z}^m \cap Q_N.$$

DEFINITION 5. A symbol of the operator $K_{d,N}$ is called the function $\sigma_{d,N}(\tilde{\xi})$ of a discrete variable $\tilde{\xi} \in \hbar\mathbb{Z}^m \cap Q_N$ defined by the formula

$$\sigma_{d,N}(\tilde{\xi}) = \sum_{\tilde{x} \in h\mathbb{Z}^m \cap Q_N} e^{i\tilde{x}\cdot\tilde{\xi}} K_{d,N}(\tilde{x}) h^m, \quad \tilde{\xi} \in \hbar\mathbb{Z}^m \cap Q_N.$$

3 Comparison between infinite and finite discrete operators

Theorem 2. *If the operator K_d is invertible in the space l^2 then the operator $K_{d,N}$ is invertible in the space l^2_N for enough large N.*

Proof. Let the function

$$\sum_{\tilde{x} \in Q^d_N} K_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m$$

is a segment of the Fourier series

$$\sum_{\tilde{x} \in h\mathbb{Z}^m} K_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m$$

and according our assumptions this is continuous function on $\hbar\mathbb{T}^m$. Therefore values of the partial sum coincide with values of $\sigma_{d,N}$ in points $\tilde{\xi} \in R^d_N \equiv \hbar\mathbb{Z}^m \cap Q_N$. Besides these partial sums are continuous functions on $\hbar\mathbb{T}^m$. □

3.1 Some auxiliary results

Lemma 1. *The norm of operator $K_d; l^2 \to l^2$ doesn't depend on h*

For the proof see [19].

Lemma 2. *The norm of operator $K_{d,N} : l^2_N \to l^2_N$ doesn't depend on N, h.*

Proof. Using the Theorem 2 and the property that the norm of the operator $K_{d,N}$ is equivalent to $\max\limits_{\xi \in \hbar\mathbb{Z}^m \cap Q_N} |\sigma_{d,N}(\xi)|$ (see also [10]) we obtain the required assertion. □

4 Correlation between N, h and location of $\tilde{x}$

The proved Theorem 1 is a very rough estimate. It is possible to obtain more exact error estimate for finite discrete solution taking into account a location of the point $\tilde{x}$ and relations between parameters N and h.

We consider three types of equations

$$Ku = v, \tag{4.1}$$
$$K_d u_d = P_d v \equiv v_d, \tag{4.2}$$

where P_d is a restriction operator which given continuous function v defined on $\mathbb{R}^m$ maps to a collection of its values on $\mathbb{Z}^m$, and

$$K_{d,N} u_{d,N} = v_{d,N} \equiv P_N v_d \tag{4.3}$$

and would like to have an estimate for nearness of their solutions.

Let us denote by $r(\tilde{x})$ the distance between $\tilde{x} \in h\mathbb{Z}^m \cap Q_N$.

Theorem 3. *Let $v_d \in C_h(\alpha, \beta)$. Then $\forall \tilde{x} \in \mathbb{Z}^m \cap Q_N$ the following estimate*

$$|u_d(\tilde{x}) - u_{d,N}(\tilde{x})| \leq c_1 \begin{cases} N^{\alpha-\beta} \ln(1 + c_2 N/h), & \text{if } r(\tilde{x}) \sim N^{-1}, \\ N^{\alpha-\beta}, & \text{in other cases} \end{cases}$$

holds, c_1, c_2 are constants non-depending on h, N.

Proof. Assuming N is enough large so that both operators K_d and $K_{d,N}$ are invertible in spaces l^2 and l^2_N respectively let us consider the difference

$$u_d(\tilde{x}) - u_{d,N}(\tilde{x}) = (K_d^{-1} v_d)(\tilde{x}) - (K_{d,N}^{-1} v_{d,N})(\tilde{x}) = \big((K_d^{-1} v_d)(\tilde{x}) \\ - (K_d^{-1} v_{d,N})(\tilde{x})\big) + \big((K_d^{-1} v_{d,N})(\tilde{x}) - (K_{d,N}^{-1} v_{d,N})(\tilde{x})\big) = I_1 + I_2,$$

where

$$I_1 = (K_d^{-1} v_d)(\tilde{x}) - (K_d^{-1} v_{d,N})(\tilde{x}),$$
$$I_2 = (K_d^{-1} v_{d,N})(\tilde{x}) - (K_{d,N}^{-1} v_{d,N})(\tilde{x}).$$

We'll consider summands separately and give pointwise estimates for them. For

$$I_2(\tilde{x}) = (K_d^{-1} v_{d,N})(\tilde{x}) - (K_{d,N}^{-1} v_{d,N})(\tilde{x})$$

we have $I_2(\tilde{x}) = 0$ because this difference is not zero for points from $h\mathbb{Z}^m \setminus Q_N$ only, so we need to estimate $I_1(\tilde{x})$ only.

First we need to say some words on a structure of operators K_d^{-1} and $K_{d,N}^{-1}$ which we have constructed for operators $K_d : l^2 \to l^2$ and $K_{d,N} : l^2_N \to l^2_N$. Lemma 1 [19] implies that the norm of the operator K_d doesn't depend on h. Also we have an analogue assertion for the operator $K_{d,N}$ (Lemma 2). Moreover the operator K_d^{-1} is generated by Calderon–Zygmund operator with symbol $\sigma^{-1}(\xi)$ and corresponding kernel $K^{-1}(x)$, so that the kernel $K_d^{-1}(\tilde{x})$ of the discrete operator K_d^{-1} is a restriction of the kernel $K^{-1}(x)$ on the lattice $h\mathbb{Z}^m$.

Further the operator $K_{d,N}^{-1}$ is constructed by the same way. We take the discrete kernel $K_d^{-1}(\tilde{x})$ then we take its restriction on Q_N and a periodic continuation on a whole $h\mathbb{Z}^m$. A symbol of a such operator will be $\sigma_{d,N}^{-1}(\tilde{\xi})$, and we conserve all required properties.

Let us start to estimate $I_1(\tilde{x})$.

$$\begin{aligned} I_1(\tilde{x}) &= \sum_{\tilde{y}\in h\mathbb{Z}^m} K_d^{-1}(\tilde{x}-\tilde{y})\left[v_d(\tilde{y}) - v_{d,N}(\tilde{y})\right] h^m \\ &= \sum_{\tilde{y}\in h\mathbb{Z}^m\setminus Q_N} K_d^{-1}(\tilde{x}-\tilde{y}) v_d(\tilde{y}) h^m \end{aligned}$$

and we need to estimate the last sum only.

1) $r(\tilde{x}) \sim N^{-1}$. Here taking into account that $|\tilde{x}-\tilde{y}| \geq h$ we obtain

$$|\tilde{x}-\tilde{y}| \sim |\tilde{x}-\tilde{y}| + h.$$

We'll represent $I_1(\tilde{x})$ as a sum $I_1(\tilde{x}) = I_{11}(\tilde{x}) + I_{12}(\tilde{x})$, where

$$\begin{aligned} I_{11}(\tilde{x}) &= \sum_{\tilde{y}\in A\cap(h\mathbb{Z}^m\setminus Q_N)} K_d^{-1}(\tilde{x}-\tilde{y}) v_d(\tilde{y}) h^m, \\ I_{12}(\tilde{x}) &= \sum_{\tilde{y}\in B\cap(h\mathbb{Z}^m\setminus Q_N)} K_d^{-1}(\tilde{x}-\tilde{y}) v_d(\tilde{y}) h^m \end{aligned}$$

and for $\tilde{y} \in A\cap(h\mathbb{Z}^m\setminus Q_N)$ we have $r(\tilde{y}) \sim N^{-1}$, for $\tilde{y} \in B\cap(h\mathbb{Z}^m\setminus Q_N)$ we assume that $r(\tilde{y}) \gg N^{-1}$. Then

$$|I_{11}(\tilde{x})| \leq \sum_{\tilde{y}\in A\cap(h\mathbb{Z}^m\setminus Q_N)} (|\tilde{x}-\tilde{y}|+h)^{-m}(1+|\tilde{y}|)^{\alpha-\beta} h^m$$

and for enough small h we have

$$|I_{11}(\tilde{x})| \leq c\int_{D_N(\tilde{x})} (|\tilde{x}-y|+h)^{-m}(1+|y|)^{\alpha-\beta} dy,$$

where $D_N(\tilde{x})$ is a ball with a center in $\tilde{x}$ and radius $\sim N$. Using spherical coordinates with a center in $\tilde{x}$ we obtain

$$|I_{11}(\tilde{x})| \leq c_1 N^{\alpha-\beta} \int_0^{c_2 N} \frac{dt}{t+h}$$

and thus

$$|I_{11}(\tilde{x})| \leq c_1 N^{\alpha-\beta} \ln(1 + c_2 N/h).$$

For $I_{12}(\tilde{x})$ we use another estimate. We write

$$\begin{aligned} |I_{12}(\tilde{x})| &\leq \sum_{\tilde{y}\in B\cap(h\mathbb{Z}^m\setminus Q_N)} (|\tilde{x}-\tilde{y}|+h)^{-m}(1+|\tilde{y}|)^{\alpha-\beta} h^m \\ &= \sum_{\tilde{y}\in B\cap(h\mathbb{Z}^m\setminus Q_N)\cap\{|\tilde{y}|\leq c|\tilde{x}|\}} (|\tilde{x}-\tilde{y}|+h)^{-m}(1+|\tilde{y}|)^{\alpha-\beta} h^m \end{aligned}$$

$$+ \sum_{\tilde{y} \in B \cap (h\mathbb{Z}^m \setminus Q_N) \cap \{|\tilde{y}| > c|\tilde{x}|\}} (|\tilde{x} - \tilde{y}| + h)^{-m} (1 + |\tilde{y}|)^{\alpha - \beta} h^m$$
$$= I_{121}(\tilde{x}) + I_{122}(\tilde{x}).$$

For $I_{121}(\tilde{x})$ we have $|\tilde{x} - \tilde{y}| \sim |\tilde{x}|$, and

$$|I_{121}(\tilde{x})| \leq c|\tilde{x}|^{-m} \int\limits_{|\tilde{x}| \leq |y| \leq c|\tilde{x}|} (1 + |y|)^{\alpha - \beta} dy \sim |\tilde{x}|^{\alpha - \beta},$$

consequently $|I_{121}(\tilde{x})| \leq cN^{\alpha - \beta}$. For $I_{122}(\tilde{x})$ we have $|\tilde{x} - \tilde{y}| \sim |\tilde{y}|$, and then

$$|I_{122}(\tilde{x})| \leq c \int_{|y| > N} (1 + |y|)^{\alpha - \beta - m} dy \sim |\tilde{N}|^{\alpha - \beta}.$$

2) $r(\tilde{x}) \sim N$. Here we have $|\tilde{x} - \tilde{y}| \sim N$. Thus

$$|I_1(\tilde{x})| = \Big| \sum_{\tilde{y} \in h\mathbb{Z}^m \setminus Q_N} K_d^{-1}(\tilde{x} - \tilde{y}) v_d(\tilde{y}) h^m \Big| \leq \sum_{\tilde{y} \in h\mathbb{Z}^m \setminus Q_N} |K_d^{-1}(\tilde{x} - \tilde{y})| |v_d(\tilde{y})| h^m$$
$$\leq cN^{-m} \sum_{\tilde{y} \in h\mathbb{Z}^m \setminus Q_N} (1 + |\tilde{y}|)^{\alpha - \beta} h^m.$$

Since we are interested in small h the last sum can be dominated by the following integral

$$\int_{\mathbb{R}^m \setminus Q_N} |y|^{\alpha - \beta} dy$$

and calculations with spherical coordinates give the estimate for $\beta - \alpha > 0$

$$|I_1(\tilde{x})| \leq cN^{\alpha - \beta}.$$

3) $r(\tilde{x}) \sim 1$. So we have the following estimate

$$|I_1(\tilde{x})| = \Big| \sum_{\tilde{y} \in h\mathbb{Z}^m \setminus Q_N} K_d^{-1}(\tilde{x} - \tilde{y}) v_d(\tilde{y}) h^m \Big| \leq \sum_{\tilde{y} \in h\mathbb{Z}^m \setminus Q_N} |K_d^{-1}(\tilde{x} - \tilde{y})| |v_d(\tilde{y})| h^m$$
$$\leq c \sum_{\tilde{y} \in h\mathbb{Z}^m \setminus Q_N} (1 + |\tilde{y}|)^{\alpha - \beta - m} h^m,$$

because $|\tilde{x} - \tilde{y}| \sim 1 + |\tilde{y}|$. Since we are interested in small h the last sum can be dominated by the following integral

$$\int_{\mathbb{R}^m \setminus Q_N} |y|^{\alpha - \beta - m} dy$$

and calculations with spherical coordinates give the estimate for $\beta - \alpha > 0$

$$|I_1(\tilde{x})| \leq cN^{\alpha - \beta}.$$

$\square$

5 Conclusions

Our considerations give a certain algorithm for solving a simplest singular integral equation in a whole space $\mathbb{R}^m$, and also in $\mathbb{R}^m_+$ taking into account authors' conclusions in [21,22]. The error estimate for finite discrete solutions shows that varying N, h we can obtain a necessary sharpness. Collecting all authors' results [18, 19, 20, 21, 22] related to equations (4.1), (4.2), (4.3) we conclude that there is a certain correspondence between solvability of these equations and their solutions.

References

[1] S. Abdullaev. Multidimensional singular integral equations in Holder spaces with a weight degenerating on a non-compact set. *Doklady Akademii Nauk SSSR*, **308**(6):1289–1292, 1989.

[2] A. Böttcher and B. Silbermann. *Analysis of Toeplitz Operators.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006. https://doi.org/10.1007/3-540-32436-4.

[3] V. Didenko and B. Silbermann. *Approximation of Additive Convolution-like Operators. Real C^*-algebra Approach.* Birkhauser, Basel, 2008.

[4] F.D. Gakhov. *Boundary Value Problems.* International Series of Monographs in Pure and Applied Mathematics. Pergamon, Massachusetts, Palo Alto, London, 1966.

[5] I.C. Gohberg and I.A. Feldman. *Convolution Equations and Projection Methods for Their Solution (Translations of Mathematical Monographs).* The American Mathematical Society, Providence, 1974.

[6] R. Hagen, S. Roch and B. Silbermann. *C^*-algebras and Numerical Analysis.* Pure and applied mathematics. A series of Monographs and textbooks. Marcel Dekker, New York, 2001.

[7] I. Lifanov, L. Poltavskii and G. Vainikko. *Hypersingular Integral Equations and their Applications.* Chapman & Hall/CRC, Boca Raton, 2004.

[8] I.K. Lifanov. *Singular Integral Equations and Discrete Vortices.* VSP, Utrecht, The Netherlands, 1996.

[9] I.K. Lifanov and L.N. Poltavskii. Pseudodifference operators and uniform convergence of divided differences. *Sbornik: Mathematics*, **193**(2):205–230, 2002. https://doi.org/10.1070/SM2002v193n02ABEH000627.

[10] S.G. Mikhlin and S. Prössdorf. *Singular Integral Operators.* Springer-Verlag, Berlin, 1986.

[11] N.I. Muskhelishvili. *Singular Integral Equations.* Springer Netherlands, 1958. https://doi.org/10.1007/978-94-009-9994-7.

[12] A. Pedas and E. Tamme. Spline collocation method for integro-differential equations with weakly singular kernels. *Journal of Computational and Applied Mathematics*, **197**(1):253–269, 2006. https://doi.org/10.1016/j.cam.2005.07.035.

[13] A. Pedas and E. Tamme. A discrete collocation method for Fredholm integro-differential equations with weakly singular kernels. *Applied Numerical Mathematics*, **61**(6):738–751, 2011. https://doi.org/10.1016/j.apnum.2011.01.006.

[14] S. Prössdorf and B. Silbermann. *Numerical Analysis for Integral and Related Operator Equations*, volume 52 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel, 1991.

[15] J. Saranen and G. Vainikko. *Periodic Integral and Pseudodifferential Equations with Numerical Approximation.* Springer Monographs in Mathematics. Springer, Berlin, 2002. https://doi.org/10.1007/978-3-662-04796-5.

[16] E. Vainikko and G. Vainikko. A spline product quasi-interpolation method for weakly singular Fredholm integral equations. *SIAM Journal on Numerical Analysis*, **46**(4):1799–1820, 2008. https://doi.org/10.1137/070693308.

[17] G. Vainikko. *Multidimensional Weakly Singular Integral Equations*, volume 1549 of *Lecture Notes in Mathematics.* Springer, Berlin, Heidelberg, 1993. https://doi.org/10.1007/BFb0088979.

[18] A.V. Vasil'ev and V. B. Vasil'ev. Periodic Riemann problem and discrete convolution equations. *Differential Equations*, **51**(5):652–660, 2015. https://doi.org/10.1134/S0012266115050080.

[19] A.V. Vasil'ev and V.B. Vasil'ev. On the solvability of certain discrete equations and related estimates of discrete operators. *Doklady Mathematics*, **92**(2):585–589, 2015. https://doi.org/10.1134/S1064562415050312.

[20] A.V. Vasilyev and V.B. Vasilyev. Numerical analysis for some singular integral equations. *Neural, Parallel & Scientific Computations*, **20**(3/4):313–326, 2012.

[21] A.V. Vasilyev and V.B. Vasilyev. Discrete singular operators and equations in a half-space. *Azerbaijan Journal of Mathematics*, **3**(1):84–93, 2013.

[22] A.V. Vasilyev and V.B. Vasilyev. Discrete singular integrals in a half-space. In V.V. Mityushev and M.V. Ruzhansky(Eds.), *Current Trends in Analysis and Its Applications: Proceedings of the 9th ISAAC Congress, Kraków* 2013, pp. 663–670, Cham, 2015. Springer International Publishing. https://doi.org/10.1007/978-3-319-12577-0_72.