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Fujita Critical Curve for a Coupled Diffusion System with Inhomogeneous Neumann Boundary Conditions*

Runmei Du and Minghao Guo

Changchun University of Technology
School of Basic Science, 130012 Changchun, China
E-mail(*corresp.*): mhguo@ccut.edu.cn
E-mail: durm_dudu@163.com

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Abstract. In this paper, we establish the blow-up theorems of Fujita type for a class of exterior problems of nonlinear diffusion equations subject to inhomogeneous Neumann boundary conditions. The critical Fujita exponents are determined and it is shown that the critical curve belongs to the blow-up case under any nontrivial initial data.

Keywords: Fujita critical curve, inhomogeneous term, global existence, blow-up, diffusion equations.

AMS Subject Classification: 35K65; 35B33.

1 Introduction

In this paper, we consider the critical curve of the following coupled diffusion system with inhomogeneous Neumann boundary conditions

$$\begin{cases} u_t = \Delta u^m + v^p, \quad v_t = \Delta v^n + u^q, & (x,t) \in D^c \times (0,+\infty), \\ \frac{\partial u^m}{\partial \nu}(x,t) = f_1(x), \quad \frac{\partial v^n}{\partial \nu}(x,t) = f_2(x), & (x,t) \in \partial D \times (0,+\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in D^c, \end{cases} \tag{1.1}$$

where $m, n > 0$, $p > \max\{1, n\}$, $q > \max\{1, m\}$, D is a bounded open domain in $\mathbb{R}^N$ with smooth boundary, $D^c = \mathbb{R}^N \setminus \overline{D}$, ν is the inward unit normal to ∂D. $u_0(x), v_0(x)$ are nonnegative continuous functions in D^c, $f_1(x), f_2(x)$ are nontrivial nonnegative continuous functions on ∂D, and $u_0(x) = f_1(x), v_0(x) = f_2(x)$ on ∂D.

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Study on critical exponents began from 1966 by Fujita in the work [6]. He considered the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $u_0 \geq 0$. It was shown that the problem has no non-trivial global solutions if $1 < p < p_c = 1 + 2/N$, whereas if $p > p_c$, there exist both global (with small initial data) and non-global (with large initial data) solutions. We call p_c the critical Fujita exponent. Later, it was proved by Hayakawa [7] and Weissler [10] that $p = p_c$ belongs to the blow-up case. From then on, there have been a lot of works on the critical exponents of Fujita type for various nonlinear evolution equations and systems [1-18]. Among these, Zeng [12] investigated the blow-up theorems of Fujita type for the following inhomogeneous problems

$$\begin{cases} u_t = \Delta u^m + u^p + f(x), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $m > 0, p > \max\{1, m\}$, $f(x)$ and $u_0(x)$ are nonnegative continuous functions in $\mathbb{R}^N$ with $f(x) \not\equiv 0$, and

$$\begin{cases} u_t = \Delta u^m + u^p & (x,t) \in D^c \times (0, +\infty), \\ \frac{\partial u^m}{\partial \nu}(x,t) = f(x), & (x,t) \in \partial D \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in D^c, \end{cases}$$

where $m > 0, p > \max\{1, m\}$, $f(x)$ is a nontrivial nonnegative continuous function on ∂D, $u_0(x)$ is a nonnegative continuous function in D^c. The author proved that $p_c = m + 2m/(N-2)_+$ is the critical exponent of both the problems.

As for coupled systems, Escobedo and Herrero [5] studied the Cauchy problem

$$\begin{cases} u_t = \Delta u + v^p, & v_t = \Delta v + u^q, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \mathbb{R}^N. \end{cases}$$

They proved that the critical Fujita curve is $(pq)_c = 1 + \frac{2}{N} \max\{p+1, q+1\}$, namely, every solution blows up in finite time if $1 < pq \leq (pq)_c$, and there exist both global and non-global solutions if $pq > (pq)_c$.

Furthermore, Yang et al. [11] studied the fast diffusion system

$$\begin{cases} u_t = \Delta u^m + v^p + f_1(x), & v_t = \Delta v^n + u^q + f_2(x), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $0 < m, n < 1$, $p, q \geq 1$, $pq > 1$, and determined the critical curve

$$(pq)_c = mn + \frac{2}{(N-2)_+} \max\{mp + mn\,, nq + mn\}.$$

The aim of this paper is to determine the critical curve to the coupled diffusion system (1.1). In the present paper, we prove the critical Fujita curve for the system (1.1) is also

$$(pq)_c = mn + \frac{2}{(N-2)_+}\max\{mp+mn\,, nq+mn\}.$$

Namely, every solution blows up in finite time if $pq < (pq)_c$, and there exist both global and non-global solutions if $pq > (pq)_c$. Further, we prove that the case $pq = (pq)_c$ belongs to the blow-up case. The method we used is similar to [1,2,3,9,11,12].

2 Main results and their proofs

First, we give the definition of the solution to the problem (1.1).

Definition 1. *A pair of functions (u,v) is said to be a weak solution of the problem (1.1), if $u\in C(0,T;L^1_{loc}(D^c))\cap C(0,T;L^q_{loc}(D^c))$, $v\in C(0,T;L^1_{loc}(D^c))$ $\cap C(0,T;L^p_{loc}(D^c))$, $\nabla u^m, \nabla v^n \in L^2(0,T;L^2_{loc}(D^c))$ and satisfy*

$$\begin{aligned}
&\int_0^\tau\int_{D^c}(u\frac{\partial\psi_1}{\partial t} - \nabla u^m\nabla\psi_1 + v^p\psi_1)dxdt + \int_0^\tau\int_{\partial D} f_1\psi_1\, dSdt \\
&\qquad\qquad = \int_{D^c} u\psi_1(x,\tau) - \int_{D^c} u_0(x)\psi_1(x,0)dx,\\
&\int_0^\tau\int_{D^c}(v\frac{\partial\psi_2}{\partial t} - \nabla v^n\nabla\psi_2 + u^q\psi_2)dxdt + \int_0^\tau\int_{\partial D} f_2\psi_2\, dSdt \\
&\qquad\qquad = \int_{D^c} v\psi_2(x,\tau) - \int_{D^c} v_0(x)\psi_2(x,0)dx
\end{aligned}$$

for any $\tau\in[0,T]$ and any compactly supported $\psi_1,\psi_2\in C^2(D^c\times[0,T])\cap C(\overline{D^c}\times [0,T])$.

The standard theory of parabolic equations ensures the well-posedness of the problem (1.1). Furthermore, we introduce the comparison principle ([4,8,13]) and the monotonicity property([12]) of the system (1.1) and omit the proof.

Lemma 1. *Let $u_1,u_2 \in C(0,T;L^1_{loc}(D^c))\cap C(0,T;L^q_{loc}(D^c))$, $v_1,v_2\in C(0,T;$ $L^1_{loc}(D^c))\cap C(0,T;L^p_{loc}(D^c))$, $\nabla u_1^m, \nabla u_2^m, \nabla v_1^n, \nabla v_2^n \in L^2(0,T;L^2_{loc}(D^c))$ and satisfy*

$$\begin{aligned}
&(u_1)_t - \Delta u_1^m - v_1^p \le (u_2)_t - \Delta u_2^m - v_2^p, && (x,t)\in D^c\times(0,+\infty),\\
&(v_1)_t - \Delta v_1^n - u_1^q \le (v_2)_t - \Delta v_2^n - u_2^q, && (x,t)\in D^c\times(0,+\infty),\\
&\frac{\partial u_1^m}{\partial\nu}(x,t) \le \frac{\partial u_2^m}{\partial\nu}(x,t),\quad \frac{\partial v_1^n}{\partial\nu}(x,t) \le \frac{\partial v_2^n}{\partial\nu}(x,t), && (x,t)\in \partial D\times(0,+\infty),\\
&u_1(x,0)\le u_2(x,0),\quad v_1(x,0)\le v_2(x,0), && x\in D^c,
\end{aligned}$$

then

$$u_1(x,t)\le u_2(x,t),\quad v_1(x,t)\le v_2(x,t),\quad (x,t)\in D^c\times(0,+\infty).$$

Lemma 2. *The nonnegative solutions of* (1.1) *with zero initial data must be monotone increasing to* t *for both components* u *and* v.

Now, we prove that the Fujita critical curve is

$$(pq)_c = mn + \frac{2}{(N-2)_+} \max\{mp + mn\,, nq + mn\}.$$

The main results are as follows.

Theorem 1. *If* $pq \le (pq)_c$, *then the system* (1.1) *has no nontrivial nonnegative global solution.*

Proof. Without loss of generality, assume $mp \ge nq$. Let $R_0 > 1$ be a fixed constant such that $D \subset B_{R_0/2}(0)$. For $R > R_0, T > 1$, choose two smooth functions $\psi_R(x)$ and $\eta_T(t)$ satisfying:

$$\begin{aligned}
&\psi_R(x) = 1,\ \ 0 \le |x| \le R/2; \quad \psi_R(x) = 0,\ |x| > R, \quad 0 \le \psi_R(x) \le 1,\ |x| > 0;\\
&\left|\frac{\partial \psi_R}{\partial x_i}(x)\right| \le \frac{C}{R}, \quad \left|\frac{\partial^2 \psi_R}{\partial x_i \partial x_j}(x)\right| \le \frac{C}{R^2}, \quad i,j = 1,2,\cdots,N;\\
&\eta_T(t) = 1,\ 0 \le t \le T/2; \quad \eta_T(t) = 0,\ t > T, \quad 0 \le \eta_T(t) \le 1,\ t > 0;\\
&|\eta_T'(t)| \le \frac{C}{T}, \quad |\eta_T''(t)| \le \frac{C}{T^2}.
\end{aligned}$$

Here and in the sequels, we use C to represent positive constants independent of R and T. Suppose by contradiction that (u,v) is a nontrivial global solution of (1.1). Define

$$\begin{aligned}
I_p &= \int_0^T \int_{B_R(0)\setminus \overline{D}} v^p(x,t)\psi_R^l(x)\eta_T^l(t)dxdt,\\
J_q &= \int_0^T \int_{B_R(0)\setminus \overline{D}} u^q(x,t)\psi_R^l(x)\eta_T^l(t)dxdt,
\end{aligned}$$

where $l > \max\{p/(p-1),\ q/(q-1)\}$. Noting the direction of ν and $\eta_T(T) = 0$, we have

$$\begin{aligned}
I_p &= \int_0^T \int_{\mathbb{R}^N\setminus \overline{D}} (u_t(x,t) - \Delta u^m(x,t))\psi_R^l(x)\eta_T^l(t)dxdt\\
&= \Big[\int_{\mathbb{R}^N\setminus \overline{D}} u(x,t)\psi_R^l(x)\eta_T^l(t)dx\Big]_0^T - \int_0^T \int_{\mathbb{R}^N} u(x,t)\psi_R^l(x)(\eta_T^l(t))'dxdt\\
&\quad - \int_0^T \int_{\mathbb{R}^N\setminus \overline{D}} u^m(x,t)\Delta(\psi_R^l(x))\eta_T^l(t)dxdt - \int_0^T \int_{\partial D} f_1(x)\eta_T^l(t)dxdt\\
&= -\int_{B_R(0)\setminus \overline{D}} u_0(x)\psi_R^l(x)dx - \int_0^T \int_{\mathbb{R}^N} u(x,t)\psi_R^l(x)(\eta_T^l(t))'dxdt\\
&\quad - \int_0^T \int_{\mathbb{R}^N\setminus \overline{D}} u^m(x,t)\Delta(\psi_R^l(x))\eta_T^l(t)dxdt - \int_0^T \int_{\partial D} f_1(x)\eta_T^l(t)dxdt.
\end{aligned}$$

Since for any $i = 1, 2, \cdots, N$,

$$\frac{\partial^2 \psi_R^l(x)}{\partial x_i^2} = \frac{\partial}{\partial x_i}\big(l\psi_R^{l-1}\frac{\partial \psi_R(x)}{\partial x_i}\big) = l\psi_R^{l-1}\frac{\partial^2 \psi_R(x)}{\partial x_i^2} + l(l-1)\psi_R^{l-2}\big(\frac{\partial \psi_R(x)}{\partial x_i}\big)^2,$$

it holds

$$\Delta \psi_R^l(x) \geq l\psi_R^{l-1}(x)\sum_{i=1}^N \frac{\partial^2 \psi_R(x)}{\partial x_i^2}.$$

Then,

$$\begin{aligned} I_p \leq & -\int_{B_R(0)\setminus\overline{D}} u_0(x)\psi_R^l(x)dx + CT^{-1}\int_{T/2}^T\int_{B_R(0)\setminus\overline{D}} u(x,t)\psi_R^l(x)\eta_T^{l-1}(t)dxdt \\ & + CR^{-2}\int_0^T\int_{B_R(0)\setminus B_{R/2}(0)} u^m(x,t)\psi_R^{l-1}(x)\eta_T^l(t)dxdt - \int_0^T\int_{\partial D} f_1(x)\eta_T^l(t)dxdt. \end{aligned} \tag{2.1}$$

Using Hölder's inequality and noting $l > \max\{p/(p-1), q/(q-1)\}$, we have

$$\begin{aligned} I_p \leq & -\int_{B_R(0)\setminus\overline{D}} u_0(x)\psi_R^l(x)dx + CT^{-1}J_q^{1/q}\Big(\int_{T/2}^T\int_{B_R(0)\setminus\overline{D}} dxdt\Big)^{1-1/q} \\ & + CR^{-2}J_q^{m/q}\Big(\int_{T/2}^T\int_{B_R(0)\setminus B_{R/2}(0)} dxdt\Big)^{1-m/q} - \int_0^T\int_{\partial D} f_1(x)\eta_T^l(t)dxdt. \end{aligned} \tag{2.2}$$

Denote

$$\delta = \min\Big\{\int_{\partial D} f_1(x)\,dx, \int_{\partial D} f_2(x)\,dx\Big\}.$$

Then

$$\int_0^T\int_{\partial D} f_i(x)\eta_T^l(t)dxdt \geq \int_0^{T/2}\int_{\partial D} f_i(x)dxdt \geq \frac{\delta}{2}T, \quad i = 1, 2. \tag{2.3}$$

Together with (2.2), we have

$$\begin{aligned} I_p + \int_{B_R(0)\setminus\overline{D}} u_0(x)\psi_R^l(x)dx + \frac{\delta}{2}T \leq & CT^{-1/q}R^{N(q-1)/q}J_q^{1/q} \\ & + CT^{1-m/q}R^{N(q-m)/q-2}J_q^{m/q}. \end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned} J_q + \int_{B_R(0)\setminus\overline{D}} v_0(x)\psi_R^l(x)dx + \frac{\delta}{2}T \leq & \\ & CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} + CT^{1-n/p}R^{N(p-n)/p-2}I_p^{n/p}. \end{aligned} \tag{2.5}$$

Consequently, we can obtain

$$\begin{aligned}
I_p &+ \int_{B_R(0)\setminus\overline{D}} u_0(x)\psi_R^l(x)dx + \frac{\delta}{2}T \le CT^{-1/q}R^{N(q-1)/q} \\
&\times \Big(CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} + CT^{1-n/p}R^{N(p-n)/p-2}I_p^{n/p}\Big)^{1/q} + CT^{1-m/q} \\
&\times R^{N(q-m)/q-2}\Big(CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} + CT^{1-n/p}R^{N(p-n)/p-2}I_p^{n/p}\Big)^{m/q} \\
\le{}& CT^{-(p+1)/(pq)}R^{N(pq-1)/(pq)}I_p^{1/(pq)} + CT^{-n/(pq)}R^{N(pq-n)/(pq)-2/q}I_p^{n/(pq)} \\
&+ CT^{(pq-mp-m)/(pq)}R^{N(pq-m)/(pq)-2}I_p^{m/(pq)} \\
&+ CT^{(pq-mn)/(pq)}R^{N(pq-mn)/(pq)-2m/q-2}I_p^{mn/(pq)}.
\end{aligned}$$

As $p > \max\{1, n\}$, $q > \max\{1, m\}$, by Young's inequality, we have

$$\begin{aligned}
I_p &+ \int_{B_R(0)\setminus\overline{D}} u_0(x)\psi_R^l(x)dx + \frac{\delta}{2}T \le \frac{1}{8}I_p + CT^{-(p+1)/(pq-1)}R^N + \frac{1}{8}I_p \\
&+ CT^{-n/(pq-n)}R^{N-2p/(pq-n)} + \frac{1}{8}I_p + CT^{(pq-mp-m)/(pq-m)}R^{N-2pq/(pq-m)} \\
&+ \frac{1}{8}I_p + CTR^{N-2(mp+pq)/(pq-mn)}.
\end{aligned} \tag{2.6}$$

Hence,

$$\begin{aligned}
\frac{\delta}{2} \le{}& CT^{-(p+1)/(pq-1)-1}R^N + CT^{-n/(pq-n)-1}R^{N-2p/(pq-n)} \\
&+ CT^{-mp/(pq-m)}R^{N-2pq/(pq-m)} + CR^{N-2(mp+pq)/(pq-mn)}.
\end{aligned} \tag{2.7}$$

If $pq < (pq)_c$, then $N - \frac{2p(q+m)}{pq-mn} < 0$. Let R be such that $CR^{N-\frac{2p(q+m)}{pq-mn}} < \delta/4$. For the fixed R, we can choose T such that

$$\begin{aligned}
CT^{-(p+1)/(pq-1)-1}R^N &+ CT^{-n/(pq-n)-1}R^{N-2p/(pq-n)} \\
&+ CT^{-mp/(pq-m)}R^{N-2pq/(pq-m)} < \frac{\delta}{4},
\end{aligned} \tag{2.8}$$

leading to a contradiction.

If $pq = (pq)_c$, then $N - \frac{2p(q+m)}{pq-mn} = 0$. From Lemma 1, it suffices to prove the blow-up of (u, v) with $u_0 = v_0 = 0$. By Lemma 2, $u_t, v_t \ge 0$. Choose $T > 0$ large such that the sum of first three terms in right hand side of (2.7) is smaller than 1. It follows from (2.6) that for fixed $R > R_0$,

$$\int_{\frac{T}{4}}^{\frac{T}{2}}\int_{B_{R/2}} v^p(x,t)dxdt \le CT$$

and consequently,

$$\inf_{t\in(\frac{T}{4},\frac{T}{2})}\int_{B_{R/2}} v^p(x,t)dxdt \le C.$$

Due to $v_t \geq 0$ and the arbitrary of T, we have $\int_{B_{R/2}} v^p(x,t)dx \leq C$ for $t > 0$ and hence

$$I_R^\infty := \lim_{t\to\infty} \int_{B_{\frac{R}{2}}} v^p(x,t)dx \leq C.$$

Since I_R^∞ is increasing with R, $\lim_{R\to\infty} I_R^\infty$ exists. So, for any $\varepsilon > 0$, there exists R_1, such that for $R \geq R_1$,

$$\int_{B_R(0)\setminus B_{R/2}(0)} v^p(x,t)dx < \varepsilon, \quad t > 0. \tag{2.9}$$

Similar to (2.1), we get

$$J_q \leq -\int_{B_R(0)\setminus\overline{D}} v_0(x)\psi_R^l(x)dx + CT^{-1}\int_{T/2}^T\int_{B_R(0)\setminus\overline{D}} v(x,t)\psi_R^l(x)\eta_T^{l-1}(t)dxdt$$
$$+ CR^{-2}\int_0^T\int_{B_R(0)\setminus B_{R/2}(0)} v^n(x,t)\psi_R^{l-1}(x)\eta_T^l(t)dxdt - \int_0^T\int_{\partial D} f_2(x)\eta_T^l(t)dxdt.$$

By Hölder inequality, (2.3) and (2.9), we have

$$\begin{aligned} J_q + \int_{B_R(0)\setminus\overline{D}} v_0(x)\psi_R^l(x)dx + \frac{\delta}{2}T &\leq CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} \\ &+ CT^{1-n/p}R^{N(p-n)/p-2}\Big(\int_0^T\int_{B_R(0)\setminus B_{R/2}(0)} v^p\psi_R^l(x)\eta_T^l(t)dxdt\Big)^{n/p} \\ &\leq CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} + CTR^{N(p-n)/p-2}\varepsilon^{n/p}. \end{aligned} \tag{2.10}$$

From (2.4), (2.5), (2.10) and $pq = (pq)_c$, we obtain

$$\begin{aligned} &I_p + \int_{B_R(0)\setminus\overline{D}} u_0(x)\psi_R^l(x)dx + \frac{\delta}{2}T \\ &\leq CT^{-1/q}R^{N(q-1)/q}\Big(CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} + CT^{1-n/p}R^{N(p-n)/p-2}I_p^{n/p}\Big)^{1/q} \\ &+ CT^{1-m/q}R^{N(q-m)/q-2}\Big(CT^{-1/p}R^{N(p-1)/p}I_p^{1/p} + CTR^{N(p-n)/p-2}\varepsilon^{n/p}\Big)^{m/q} \\ &\leq CT^{-(p+1)/(pq)}R^{N(pq-1)/(pq)}I_p^{1/(pq)} + CT^{-n/(pq)}R^{N(pq-n)/(pq)-2/q}I_p^{n/(pq)} \\ &+ CT^{(pq-mp-m)/(pq)}R^{N(pq-m)/(pq)-2}I_p^{m/(pq)} + CT\varepsilon^{mn/(pq)}. \end{aligned}$$

By Young inequality, we have

$$\begin{aligned} I_p + \delta T \leq \frac{1}{6}I_p &+ CT^{-(p+1)/(pq-1)}R^N + \frac{1}{6}I_p + CT^{-n/(pq-n)}R^{N-2p/(pq-n)} \\ &+ \frac{1}{6}I_p + CT^{(pq-mp-m)/(pq-m)}R^{N-2pq/(pq-m)} + CT\varepsilon^{mn/(pq)}, \end{aligned}$$

which implies

$$\begin{aligned} \frac{\delta}{2} \leq &CT^{-(p+1)/(pq-1)-1}R^N + CT^{-n/(pq-n)-1}R^{N-2p/(pq-n)} \\ &+ CT^{-mp/(pq-m)}R^{N-2pq/(pq-m)} + C\varepsilon^{mn/(pq)}. \end{aligned}$$

Take $\varepsilon < (\frac{\delta}{4C})^{pq/(mn)}$, there exists R_2, such that for a fixed $R > R_2$, (2.9) is valid. Then choose T large enough such that (2.8) holds, a contradiction. □

Theorem 2. *If $pq > (pq)_c$, then there exist solutions of the system* (1.1) *blow up in finite time.*

Proof. Suppose by contradiction that every solution (u, v) of the system (1.1) is a global solution. Similar to the proof of Theorem 1, we assume $mp \geq nq$. Then, for any fixed $R > R_0$, $T > 1$, we have (2.6). Thus,

$$\begin{aligned}&\int_{B_{R/2}(0)\setminus B_{R_0/2}(0)} u_0(x)dx\\ \leq &CT^{-(p+1)/(pq-1)}R^N + CT^{-n/(pq-n)}R^{N-2p/(pq-n)}\\ &+ CT^{(pq-mp-m)/(pq-m)}R^{N-2pq/(pq-m)} + CTR^{N-2(mp+pq)/(pq-mn)}. \end{aligned} \tag{2.11}$$

Let κ be large enough such that

$$\begin{aligned}&\kappa\int_{B_{R/2}(0)\setminus B_{R_0/2}(0)} 1dx\\ > &CT^{-(p+1)/(pq-1)}R^N + CT^{-n/(pq-n)}R^{N-2p/(pq-n)}\\ &+ CT^{(pq-mp-m)/(pq-m)}R^{N-2pq/(pq-m)} + CTR^{N-2(mp+pq)/(pq-mn)}.\end{aligned}$$

Then for $u_0(x) > \kappa\psi_R(x)$, we have

$$\begin{aligned}&\int_{B_{R/2}(0)\setminus B_{R_0/2}(0)} u_0(x)dx > \kappa\int_{B_{R/2}(0)\setminus B_{R_0/2}(0)} 1dx\\ > &CT^{-(p+1)/(pq-1)}R^N + CT^{-n/(pq-n)}R^{N-2p/(pq-n)}\\ &+ CT^{(pq-mp-m)/(pq-m)}R^{N-2pq/(pq-m)} + CTR^{N-2(mp+pq)/(pq-mn)},\end{aligned}$$

which contradicts (2.11). □

Theorem 3. *If $pq > (pq)_c$, then there exist global solutions of the system* (1.1).

Proof. Note $pq > (pq)_c$ implies $N - \max\{\alpha_1 m + 2, \alpha_2 n + 2\} > 0$. Define

$$\begin{aligned}M_1 &= \Big(\alpha_1 m(N - \alpha_1 m - 2)\Big)^{n/(pq-mn)}\Big(\alpha_2 n(N - \alpha_2 n - 2)\Big)^{p/(pq-mn)},\\ M_2 &= \Big(\alpha_1 m(N - \alpha_1 m - 2)\Big)^{q/(pq-mn)}\Big(\alpha_2 n(N - \alpha_2 n - 2)\Big)^{m/(pq-mn)}.\end{aligned}$$

Let

$$U(x) = M_1(1 + |x|^2)^{-\alpha_1/2}, \quad V(x) = M_2(1 + |x|^2)^{-\alpha_2/2}, \quad x \in D^c,$$

where

$$\alpha_1 = \frac{2(p+n)}{pq - mn}, \quad \alpha_2 = \frac{2(q+m)}{pq - mn}.$$

Then (U,V) satisfy

$$\begin{cases} -\Delta U^m - V^p = M_1^m \alpha_1 m(\alpha_1 m+2)(1+|x|^2)^{-m\alpha_1/2-2}, & x \in D^c, \\ -\Delta V^n - U^q = M_2^n \alpha_2 n(\alpha_2 n+2)(1+|x|^2)^{-n\alpha_2/2-2}, & x \in D^c. \end{cases}$$

Now for $f_1(x), f_2(x)$, $u_0(x), v_0(x)$ small enough such that

$$f_1(x) \le \frac{\partial U^m}{\partial \nu}, \quad f_2(x) \le \frac{\partial V^n}{\partial \nu}, \quad x \in \partial D,$$

and

$$u_0(x) \le U(x), \ v_0(x) \le V(x), \quad x \in D^c,$$

$(U(x), V(x))$ is a global supersolution of (1.1) by Lemma 1. □

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