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New Type of Difference Sequence Spaces of Fuzzy Real Numbers

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Abstract. In this paper we introduce the natation difference operator $\Delta_m (m \geq 0,$ an integer) for studying properties of some sequence spaces. We define the sequence spaces $\ell_\infty^F(\Delta_m)$, $c^F(\Delta_m)$, $c_0^F(\Delta_m)$ and investigate their properties like solidness, convergence free, symmetricity, completeness.

Key words: Fuzzy sequences, difference sequence space, solid space, symmetric space, convergence free, completeness.

1 Introduction

The concept of fuzzy set was introduced by Zadeh [11]. Bounded and convergent sequences of fuzzy numbers were studied by Matloka [6], where it was shown that every convergent sequence is bounded. Later, different classes of sequences of fuzzy numbers have been studied by Esi [2], Tripathy and Nanda [10], Savas [7], Fang and Hung [3], Choudhary and Tripathy [1], Tripathy and Borgohain [8].

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on the real line R. For $X, Y \in D$ we define

$$d(X,Y) = \max(|a_1 - b_1|, |a_2 - b_2|) \text{ where } X = [a_1, a_2],\ Y = [b_1, b_2].$$

It is known that (D, d) is a complete metric space.

Let $I = [0, 1]$. A fuzzy real number X is a fuzzy set on R and is a mapping $X : R \to I$ associating each real number t with its grade member ship $X(t)$.

A fuzzy real number X is called *convex* if

$$X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r)),\ \text{where } s < t < r.$$

A fuzzy real number X is called *normal* if there exists $t_0 \in R$, such that $X(t_0) = 1$. A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ and given $\varepsilon 0, X^{-1}([0, a+\varepsilon))$, is open in the usual topology of R.

The set of all *upper-semi continuous, normal, convex* fuzzy numbers is denoted by $R(I)$. The α-level set of a fuzzy real number X, for $0 < \alpha \leq 1$ denoted by X^α is defined as $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level set is the closure of strong $0-$cut.

For each $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r. \end{cases}$$

The absolute value $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkla [4])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ defines a metric on $R(I)$ (Matloka [6]). The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

In this paper we have introduced the classes of the difference sequences $c^F(\Delta_m)$, $c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ of fuzzy real numbers. We have shown that these are complete metric spaces. We have investigated different properties like solidness, symmetricity and convergence free of these spaces.

2 Definitions and Preliminaries

Throughout the article w^F, c^F, c_0^F, ℓ_∞^F denote the classes of *all, convergent, null, bounded* sequence spaces of fuzzy real numbers.

A fuzzy real valued sequence $\{X_n\}$ is said to be *convergent* to the fuzzy real number X, if for $\varepsilon > 0$, there exists $n_o \in N$ such that $\bar{d}(X_k, X) < \varepsilon$, for all $k \geq n_0$ (Matloka [6]).

A sequence space E^F is said to be *solid* (or *normal*) if $(X_k) \in E^F$ implies that $(\alpha_k X_k) \in E^F$ for all sequences of scalars (α_k) with $|a_k| \leq 1$, for all $k \in N$.

Let $K = \{k_1 < k_2 < \ldots\} \subseteq N$ and E^F be a sequence space. A *K-step space* of E^F is a sequence space $\lambda_K^{E^F} = \{(X_{k_n}) \in w^F : (X_n) \in E^F\}$.

A canonical preimage of a sequence $\{X_k\} \in \lambda_K^{E^F}$ is a sequence $\{Y_n\} \in w^F$ defined as

$$Y_n = \begin{cases} X_n, & \text{if} \quad n \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

A *canonical preimage* of a step space $\lambda_K^{E^F}$ is a set of canonical preimages of all elements in $\lambda_K^{E^F}$, i.e. Y is in canonical preimage of $\lambda_K^{E^F}$ if and only if Y is canonical preimage of some $X \in \lambda_K^{E^F}$.

A sequence space E^F is said to be *monotone* if it contains the canonical preimages of its step spaces.

A sequence space E^F is said to be *convergence free* if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $Y_k = \bar{0}$ whenever $X_k = \bar{0}$.

Kizmaz [5] defined the difference sequence space for crisp set. This concept was further generalized by Tripathy and Esi [9] as follows. Let $m \geq 0$ be an integer then $Z(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in Z\}$, for $Z = c, c_0, \ell_\infty$ where $\Delta_m x_k = x_k - x_{k+m}$, for all $k \in N$. For $m = 1$, the spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$ are studied by Kizmaz [5]. The idea of Kizmaz [5] was applied by Savas [7] for introducing the notion of difference sequences for fuzzy real numbers and study their different properties.

We introduce the following difference sequences of fuzzy real numbers of Tripathy & Esi [9] type as follows. Let $m \geq 0$ be an integer then,

$$Z(\Delta_m) = \{(X_k) \in w^F : (\Delta_m X_k) \in Z\}, \quad \text{for } Z = c^F, c_0^F, \ell_\infty^F,$$

where $\Delta_m X_k = X_k - X_{k+m}$, for all $k \in N$.

3 Main Results

Theorem 1. *The sequence spaces $c^F(\Delta_m), c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ are complete metric spaces by the metric*

$$\rho(X, Y) = \sum_{k=1}^{m} \bar{d}(X_k, Y_k) + \sup_k \bar{d}(\Delta_m X_k, \Delta_m Y_k).$$

Proof. Let (X^i) be any Cauchy sequence in $\ell_\infty^F(\Delta_m)$ where $X^i = (X_k^i) = (X_1^i, X_2^i, X_3^i, \ldots) \in \ell_\infty^F(\Delta_m)$ for each $i \in N$. Then for given $\varepsilon \geq 0$, there exist $n_0 \in N$ such that

$$\rho(X^i, Y^j) = \sum_{k=1}^{m} \bar{d}(X_k^i, Y_k^j) + \sup_k \bar{d}(\Delta_m X_k^i, \Delta_m Y_k^j) < \varepsilon, \text{ for } i, j \geq n_0. \quad (3.1)$$

Hence

$$\sum_{k=1}^{m} \bar{d}(X_k^i, X_k^j) < \varepsilon, \text{ for all } i, j \geq n_0 \text{ and } k = 1, 2, 3, \ldots m$$
$$\Longrightarrow \bar{d}(X_k^i, Y_k^j) < \varepsilon, \text{ for all } i, j \geq n_0 \text{ and } k = 1, 2, 3, \ldots m$$
$$\Longrightarrow (X_k^j) \text{ is a Cauchy sequence in } R(I) \text{ for } k = 1, 2, 3, \ldots m$$
$$\Longrightarrow (X_k^j) \text{ is a convergent sequence in } R(I) \text{ for } k = 1, 2, 3, \ldots m.$$

Let $\lim_{j\to\infty} X_k^j = X_k$ (say) for $k = 1, 2, 3, \ldots m$. Again from (3.1)

$$\bar{d}(\Delta_m X_k^i, \Delta_m X_k^j) < \varepsilon, \text{ for } \ i, j \geq n_0 \text{ for } k \in N$$
$$\Longrightarrow (\Delta_m X_k^j) \text{ is a Cauchy sequence in } R(I) \text{ for all } k \in N$$
$$\Longrightarrow (\Delta_m X_k^j) \text{ is a convergent sequence in } R(I) \text{ for all } k \in N.$$

Let $\lim_{j\to\infty} \Delta_m X_k^j = Y_k$ for each $k \in N$. Since $\lim_{j\to\infty} X_k^j = X_k$, for $k = 1, \ldots m$, then $\lim_{j\to\infty} X_k^j = X_k$ exist for each $k \in N$. For all $i \geq n_0$, we have

$$\lim_{j\to\infty} \sum_{k=1}^{m} \bar{d}(X_k^i, X_k^j) = \sum_{k=1}^{m} \bar{d}(X_k^i, X_k) < \varepsilon,$$

$$\lim_{j\to\infty} \bar{d}(\Delta_m(X_k^i, \Delta_m X_k^j) = \bar{d}(\Delta_m X_k^i, \Delta_m X_k) < \varepsilon.$$

Hence for all $i \geq n_0$, it follows that

$$\sup_k \bar{d}(\Delta_m X_k^i, \Delta_m X_k) < \varepsilon.$$

Thus for all $i, j \geq n_0$ we obtain

$$\sum_{k=1}^{m} \bar{d}(X_k^i, X_k) + \sup_k \bar{d}(\Delta_m X_k^i, \Delta_m X_k) < 2\varepsilon \quad \Longrightarrow \quad \rho(X^i, X) < 2\varepsilon,$$

i.e. $X^i \to X$, as $i \to \infty$ in $\ell_\infty^F(\Delta_m)$ follows. For $i \geq n_0$, we have

$$\sup_k \bar{d}(\Delta_m X_k, \bar{0}) \leq \sup_k \bar{d}(\Delta_m X_k, \Delta_m X_k^i) + \sup_k \bar{d}(\Delta_m X_k^i, \bar{0}) < \infty.$$

This completes the proof. □

Result 2. *The spaces $c^F(\Delta_m), c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ are not solid in general.*

Proof. The result follows from the following example:

Example 1. Consider the sequence (X_n) defined by

$$X_n(t) = \begin{cases} \dfrac{nt+n+1}{n+1}, & \text{for } -1-\dfrac{1}{n} \leq t \leq 0, \\ \dfrac{n+1-nt}{n+1}, & \text{for } 0 \leq t \leq 1+\dfrac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

For $m = 3$ we have that

$$\Delta_3 X_n(t) = \begin{cases} \dfrac{tn^2+2n^2+3nt+8n+3}{2n^2+8n+3}, & \text{for } -2-\dfrac{1}{n}-\dfrac{1}{n+3} \leq t \leq 0, \\ \dfrac{-tn^2-3tn+2n^2+8n+3}{2n^2+8n+3}, & \text{for } 0 \leq t \leq 2+\dfrac{1}{n}+\dfrac{1}{n+3}, \\ 0, & \text{otherwise.} \end{cases}$$

Now $\lim_{n\to\infty} \Delta_3 X_n = X$, where

$$X(t) = \begin{cases} \dfrac{t+2}{2}, & \text{for } -2 \leq t \leq 0, \\ \dfrac{2-t}{2}, & \text{for } 0 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(X_n) \in c^F(\Delta_3)$. Now consider the sequence of scalars (α_n) defined by

$$(\alpha_n) = \begin{cases} 1, & \text{for } n = 3k-2, \text{ for } k \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\alpha_n X_n) = \{X_1, \bar{0}, \bar{0}, X_4, \bar{0}, \bar{0}, X_7, \bar{0}, \bar{0}, X_{10}, \ldots\}$. But

$$(\Delta_3 \alpha_n X_n) = \{X_1 - X_4, \bar{0}, \bar{0}, X_4 - X_7, \bar{0}, \bar{0}, \ldots\} \notin c^F(\Delta_3).$$

Hence $c^F(\Delta_m)$ is not solid.
□

Similar examples can be constructed for the other spaces.

Result 3. *The space $c^F(\Delta_m), c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ are not symmetric in general.*

Proof. The result follows from the following example.

Example 2. Let $m = 1$, and consider the sequence $X = \{A, B, A, B, A, B, \ldots\}$, where

$$A = \begin{cases} \dfrac{t+4}{4}, & \text{for } -4 \le t \le 0, \\ \dfrac{4-t}{4}, & \text{for } 0 \le t \le 4, \\ 0, & \text{otherwise.} \end{cases}, \quad B = \begin{cases} \dfrac{t+5}{5}, & \text{for } -5 \le t \le 0, \\ \dfrac{5-t}{5}, & \text{for } 0 \le t \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Now consider the re-arrangement (Y_n) of the sequence (X_n) as

$$(Y_n) = \{A, A, B, B, A, A, \ldots\} \notin c^F(\Delta), \text{ but } (X_n) \in c^F(\Delta).$$

Hence $c^F(\Delta_m)$ is not Symmetric for any $m \in N$.
□

Similar examples can be constructed for the other spaces.

Result 4. *The spaces $c^F(\Delta_m), c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ is not convergence free.*

Proof. The result follows from the following example.

Example 3. Consider the sequence (X_n) defined as follows

$$X_n(t) = \begin{cases} \dfrac{nt+n+1}{n+1}, & \text{for } -1 - \dfrac{1}{n} \le t \le 0, \\ \dfrac{n+1-nt}{n+1}, & \text{for } 0 \le t \le 1 + \dfrac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Now we obtain

$$\Delta_3 X_n(t) = \begin{cases} \dfrac{tn^2+2n^2+3nt+8n+3}{2n^2+8n+3}, & \text{for } -2-\dfrac{1}{n}-\dfrac{1}{n+3} \le t \le 0, \\ \dfrac{-tn^2-3nt+2n^2+8n+3}{2n^2+8n+3}, & \text{for } 0 \le t \le 2+\dfrac{1}{n}+\dfrac{1}{n+3}, \\ 0, & \text{otherwise} \end{cases}$$

and $\lim_{n\to\infty} \Delta_3 X_n = X$ defined as follows

$$X(t) = \begin{cases} \dfrac{t+2}{2}, & \text{for } -2 \le t \le 0, \\ \dfrac{2-t}{2}, & \text{for } 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(X_n) \in c^F(\Delta_3)$. Now consider

$$Y_n(t) = \begin{cases} \dfrac{t+n}{n}, & \text{for } -n \le t \le 0, \\ \dfrac{n-t}{n}, & \text{for } 0 \le t \le n, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Delta_3 Y_n(t) = \begin{cases} \dfrac{t+2n+3}{2n+3}, & \text{for } -2n-3 \le t \le 0, \\ \dfrac{2n+3-t}{2n+3}, & \text{for } 0 \le t \le 2n+3, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $(Y_n) \in c^F(\Delta_3)$. Therefore the space $c^F(\Delta_3)$ is not convergence free. □

Similar examples can be constructed for other spaces.

4 Conclusions

Following the notion of difference operator Δ_m, introduced by Tripathy and Esi [7], the difference sequences $c^F(\Delta_m)$, $c_0^F(\Delta_m)$ and $\ell_\infty^F(\Delta_m)$ of fuzzy numbers have been introduced. It is shown that these classes of sequences are complete metric spaces. It is observed that these classes are neither solid nor convergence free nor symmetric. The introduced notion of difference operator can be applied for studying many other classes of sequences of fuzzy numbers.

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