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Convergence Analysis On Hybrid Projection Algorithms For Equilibrium Problems and Variational Inequality Problems*

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Abstract. In this paper, we consider an iterative method for equilibrium problems, fixed point problems and variational inequality problems in the framework of Banach space. The results presented in this paper improve and extend the corresponding results announced by many others.
Key words: Strong convergence; Equilibrium problem; Quasi-ϕ-nonexpansive mapping; Variational inequality.

1 Introduction and Preliminaries

Let E be a real Banach space with the norm $\|\cdot\|$ and C be a nonempty closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E into E^* and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$, see [9, 27] for more details.
Recall that a mapping $A : C \to E^*$ is said to be monotone

$$\langle x - y,\, Ax - Ay\rangle \geq 0, \quad \forall x, y \in C.$$

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Recall that a mapping $A : C \to E^*$ is said to be α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay,\, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Recall also that a monotone mapping A is said to be maximal if its graph $G(A) = \{(x, f) : f \in Ax\}$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping A is maximal iff for $(x, f) \in E \times E^*$ $\langle x - y,\, f - g\rangle \geq 0$ for every $(y, g) \in G(A)$ implies $f \in Ax$.

Let Q be a natural map from E into E^{**}. Then the topology for E^* induced by the topologizing family $Q(E)$ is the weak* topology of E^*. An operator A from C into E^* is said to be hemi-continuous if for all $x, y \in C$, the mapping f of $[0, 1]$ into E defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Next, we consider the following variational inequality problem for the monotone mapping $A : C \to E^*$: *find an* $u \in C$ *such that*

$$\langle Au,\, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denoted by $VI(C, A)$ the set of solutions of the variational inequality problem (1.1). From Takahashi [27], we have the following statement.

Remark 1. Let C be a nonempty convex subset of a Banach space E and let A be a monotone and hemi-continuous operator from C into E^* with $C = D(A)$. Then the set $VI(C, A)$ is closed and convex. Further, if C is compact, then $VI(C, A)$ is nonempty.

Problem (1.1) is connected with the convex minimization problem, the complementarity problem, see [11, 17, 27] for more details.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. The equilibrium problem is to find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx,\, y - x\rangle$ for all $x, y \in C$. Then $p \in EP(f)$ if and only if $\langle Tp, y - p\rangle \geq 0$ for all $y \in C$, i.e. p is a solution of the variational inequality. Numerous problems in physics, optimization, and economics can be reduced to (1.2). Some methods have been proposed to solve the equilibrium problem, see, for instance, [4, 10, 16].

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, we may assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$ we have $\lim_{t \to 0+0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{$\mathit{\Gamma}$}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit $(\mathit{\Gamma})$ is attained uniformly for $x, y \in U$. The norm of E is said to be Fréchet differentiable if, for any $x \in U$, the limit $(\mathit{\Gamma})$ is attained uniformly for all $y \in U$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\Big\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 :\ x, y \in X,\ \|x\| = 1,\ \|y\| = \tau\Big\},$$

where $\rho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The modulus of convexity of E is the function $\delta : (0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf\Big\{1 - \Big\|\frac{x + y}{2}\Big\| :\ \|x\| = \|y\| = 1,\ \|x - y\| = \epsilon\Big\}.$$

E is said to be uniformly convex if and only if $\delta(\epsilon) > 0$ for all $0 < \epsilon \le 2$. Let $p > 1$, then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\epsilon) \ge c\epsilon^p$ for all $\epsilon \in [0, 2]$. Observe that every p-uniformly convex space is uniformly convex. It is well known (see for example [29]) that

$$L^p(l^p) \text{ or } W^p_m \text{ is } \begin{cases} p - \text{uniformly convex} & \text{if } \ p \ge 2; \\ 2 - \text{uniformly convex} & \text{if } \ 1 < p \le 2. \end{cases}$$

Next, assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.3}$$

Observe that in a Hilbert space H, functional (1.3) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. Alber [1] recently introduced a generalized projection operator $\mathit{\Pi}_C$ in a real Banach space which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\mathit{\Pi}_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$ in C. That is, $\mathit{\Pi}_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{1.4}$$

Existence and uniqueness of the operator $\mathit{\Pi}_C$ follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 9, 14, 27]). In Hilbert spaces, $\mathit{\Pi}_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.5}$$

Remark 2. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.5), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [9, 27] for more details.

Let C be a closed convex subset of E, T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [21] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. A point p in C is said to be a strongly asymptotic fixed point of T [30] if C contains a sequence $\{x_n\}$ which converges strongly to p such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

In this paper, $\overline{F(T)}$ denotes the set of strongly asymptotic fixed points of the mapping T, $\widetilde{F(T)}$ denotes the set of asymptotic fixed points of the mapping T and $F(T)$ denotes the set of fixed point of the mapping T, respectively.

Definition 1. Recall that a mapping T from C into itself is said to be relatively nonexpansive [5, 6, 8, 15] if

(i) $\widetilde{F(T)} = F(T) \neq \emptyset$;

(ii) $\phi(p, Tx) \leq \phi(p, x) \quad \forall x \in C, p \in F(T)$.

The asymptotic behaviour of a relatively nonexpansive mapping was studied in [3, 5].

Definition 2. Recall that a mapping T from C into itself is said to be relatively weak nonexpansive [30] if

(i) $\overline{F(T)} = F(T) \neq \emptyset$;

(ii) $\phi(p, Tx) \leq \phi(p, x) \quad \forall x \in C, p \in F(T)$.

Definition 3. Recall that T is said to be ϕ-nonexpansive [19] if

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T).$$

Definition 4. Recall that T is said to be quasi-ϕ-nonexpansive [19] if

$$F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \ \forall\, x \in C, p \in F(T).$$

Remark 3. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively weak nonexpansive mappings and the class of relatively nonexpansive mappings which require strong restrictions: $F(T) = \widetilde{F(T)}$ and $F(T) = \overline{F(T)}$.

We have the following implications $\mathbb{A} \Longrightarrow \mathbb{B} \Longrightarrow \mathbb{C}$, where $\mathbb{A}$ denotes the class of relatively nonexpansive mappings, $\mathbb{B}$ denotes the class of relatively weak nonexpansive mappings and $\mathbb{C}$ denotes the class of quasi-ϕ-nonexpansive mappings, respectively.

Next, we give some examples of quasi-ϕ-nonexpansive mappings.

Example 1. (Qin et al. [19]). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2. (Qin et al. [19]). Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Recently, many authors studied variational inequality problems, fixed point problems and equilibrium problems by hybrid projection algorithms in the framework of Hilbert spaces and Banach spaces, respectively; see, for instance, [7, 12, 13, 18, 19, 20, 23, 25, 26, 28, 30] and the references given therein.

In 2004, Iiduka, Takahashi and Toyoda [12] introduced the following hybrid projection algorithm in a real Hilbert space:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases}$$

where $A : C \to H$ is inverse-strongly monotone mapping. They proved that the sequence $\{x_n\}$ generated by above iterative algorithm converges strongly to $P_{VI(C,A)}(x)$ provided that $VI(C,A) \neq \emptyset$, where $P_{VI(C,A)}$ is the metric projection from C onto $VI(C,A)$.

In [13], Iiduka and Takashi obtained an analogue result in the framework of Banach spaces. To be more precise, they proved the following result.

Theorem 1. *Let E be a 2-uniformly convex and uniformly smooth Banach space and let A be an operator from E into E^* which satisfies the conditions*

(1) *A is inverse-strongly monotone,*

(2) $A^{-1}(0) \neq \emptyset$.

If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, c^2\alpha/2]$ for some a with $0 < a \leq c^2\alpha/2$, then the sequence $\{x_n\}$ generated by the following process

$$\begin{cases} x_1 = x \in C, \\ y_n = J^{-1}(Jx_n - \lambda_n A x_n), \\ X_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Y_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{X_n \cap Y_n} x, \quad \forall n \geq 1, \end{cases}$$

converges strongly to $\Pi_{A^{-1}(0)}x$, where $1/c$ is the 2-uniformly convex constant of E and $\Pi_{A^{-1}(0)}$ is the generalized projection from E onto $A^{-1}(0)$.

Recently, Zegeye and Shahzad [30] improved Theorem 1 by considering the following hybrid projection algorithm:

$$\begin{cases} x_0 \in K \quad \text{chosen arbitrarily,} \\ y_n = \Pi_C[J^{-1}(Jx_n - \lambda_n Ax_n)], \ z_n = Ty_n, \\ H_0 = \{v \in K : \phi(v, z_0) \le \phi(v, y_0) \le \phi(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \le \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_0 = K, \ W_n = \{v \in W_{n-1} \cap H_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad \forall n \ge 1. \end{cases}$$

To be more precise, they obtained the following result.

Theorem 2. *Let E be a real uniformly smooth and 2-uniformly convex Banach space with dual E^*. Let K be a nonempty, closed and convex subset of E. Let $A : K \to E^*$ be a γ-inverse strongly monotone mapping and $T : K \to K$ be a relatively weak nonexpansive mapping such that $VI(K, A) \cap F(T) \neq \emptyset$. Assume that $\|Ax\| \le \|Ax - Ap\|$ for all $x \in K$ and $p \in VI(K, A)$. Let*

$$0 < a_0 \le \alpha_n \le b_0 = \gamma c^2/2,$$

where c is some constant. Then the sequence $\{x_n\}$ generated by above hybrid projection algorithm converges strongly to $\Pi_{F(T)\cap VI(K,A)}x_0$, where the operator $\Pi_{F(T)\cap VI(K,A)}$ is the generalized projection from E onto $F(T) \cap VI(K, A)$.

Very recently, Qin, Cho and Kang [19] studied the so-called equilibrium problem for a bifunction f in a Banach space which includes Takahashi and Zembayashi [28] as a special case, see [28] for more details. More precisely, they proved the following result.

Theorem 3. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $T, S : C \to C$ be two closed quasi-ϕ-nonexpansive mappings such that $F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily,} \\ C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \ \textit{such that} \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty} \alpha_n\beta_n > 0, \quad \liminf_{n\to\infty} \alpha_n\gamma_n > 0$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap F(S)\cap EP(f)} x_0$.

In this paper, motivated and inspired by the research going on in this direction, we introduce a more general hybrid projection algorithm (see below) to find a common element of the set of solutions of equilibrium problem (1.2), the set of solutions of variational inequality problems (1.1) and the set of fixed points of a quasi-ϕ-nonexpansive mapping in the framework Banach spaces. The results presented in this paper mainly improve the results of [13] and [30].

In order to prove our main results, we also need the following lemmas.

Lemma 1. [14] *Let* E *be a uniformly convex and smooth Banach space and let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* E. *If* $\phi(x_n, y_n) \to 0$ *and either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded, then* $x_n - y_n \to 0$.

Lemma 2. [1] *Let* C *be a nonempty closed convex subset of a smooth Banach space* E *and* $x \in E$. *Then,* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 3. [1] *Let* E *be a reflexive, strictly convex and smooth Banach space, let* C *be a nonempty closed convex subset of* E *and let* $x \in E$. *Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 4. [19] *Let* E *be a uniformly convex and smooth Banach space, let* C *be a closed convex subset of* E, *and let* T *be a closed and quasi-*ϕ*-nonexpansive mapping from* C *into itself. Then* $F(T)$ *is a closed convex subset of* C.

Lemma 5. [4] *Let* C *be a closed convex subset of a smooth, strictly convex, and reflexive Banach space* E, *let* f *be a bifunction from* $C \times C$ *to* R *satisfying (A1)–(A4), and let* $r > 0$ *and* $x \in E$. *Then, there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 6. *Let* C *be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space* E, *and let* f *be a bifunction from* $C \times C$ *to* R *satisfying (A1)–(A4). For* $r > 0$ *and* $x \in E$, *define a mapping* $T_r : E \to C$ *as follows:*

$$T_r x = \Big\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle, \quad \forall y \in C \Big\}.$$

Then the following hold:

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive-type mapping, i.e.,* $\forall x, y \in E$,

$$\langle T_r x - T_r, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r, Jx - Jy \rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ *is closed and convex;*

(5) T_r *is quasi-ϕ-nonexpansive.*

Proof. It follows from Lemma 5 that T_r is well-defined. From Lemma 2.8 of Takahashi and Zembayashi [28], we see that (1)–(4) hold. From [28], we also see that T_r is relatively nonexpansive. From definition of quasi-ϕ-nonexpansive mappings, we see that T_r is quasi-ϕ-nonexpansive. This completes the proof. □

Lemma 7. [28] *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$. Then*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x), \quad \forall x \in E, q \in F(T_r).$$

Lemma 8. [3] *Let E be a 2-uniformly convex Banach space. Then we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \quad \forall x, y \in E, \tag{1.6}$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

We denote by $N_C(x)$ the normal cone for C at a point $x \in C$, that is $N_C(x) := \{x^* \in E^* : \langle x - y, x^* \rangle \geq 0, \forall y \in C\}$. The following lemma is important for our main results.

Lemma 9. [22] *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemi-continuous operator of C into E. Let $Q \subset E \times E^*$ be an operator defined as follows:*

$$Qx := \begin{cases} Ax + N_C x, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then Q is maximal monotone and $Q^{-1}(0) = VI(C, A)$.

Albert [1] studied the following functional $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^* - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*.$$

From the definition of the functional V, we see that $V(x, x^*) = \phi(x, J^{-1}x^*)$.

Lemma 10. [1] *Let E be a reflexive, strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$$

2 Main Results

Theorem 4. *Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), A an α-inverse-strongly monotone mapping from C into E^* and $T : C \to C$ a closed quasi-ϕ-nonexpansive mapping. Assume that $\Omega = F(T) \cap EP(f) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = \Pi_C [J^{-1}(Jx_n - \lambda_n Ax_n)], \quad z_n = Ty_n, \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the duality mapping on E and $\{r_n\}$ is a positive number sequence such that $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$ and $0 < \beta \leq \lambda_n \leq c^2\alpha/2$, for all $n \geq 1$, where c is the constant defined by (1.6). Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

Proof. From Remark 1, Lemma 4 and Lemma 6, we see that Ω is closed and convex. First, we show that C_n is closed and convex for all $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. Since

$$\phi(v, u_k) \leq \phi(v, z_k) \iff 2\langle v, Jz_k - Ju_k\rangle \leq \|z_k\|^2 - \|u_k\|^2, \tag{2.1}$$

we have that $D_{k+1} := \{v \in C_k : \phi(v, u_k) \leq \phi(v, z_k)\}$ is closed and convex. It is easy to see that D_{k+1} is closed. Next, we show that D_{k+1} is convex. Indeed, for any $v_1, v_2 \in D_{k+1}$, we see that v_1 and $v_2 \in C_k$ and satisfy (2.1). That is,

$$2\langle v_1, Jz_k - Ju_k\rangle \leq \|z_k\|^2 - \|u_k\|^2 \tag{2.2}$$

and

$$2\langle v_2, Jz_k - Ju_k\rangle \leq \|z_k\|^2 - \|u_k\|^2. \tag{2.3}$$

For any $t \in (0.1)$, multiplying (2.2) by t and adding to (2.3) multiplied by $1-t$ yields

$$2\langle tv_1 + (1-t)v_2, Jz_k - Ju_k\rangle \leq \|z_k\|^2 - \|u_k\|^2. \tag{2.4}$$

Sin C_k is closed and convex by assumption, we have that $tv_1 + (1-t)v_2 \in C_k$, which combines with (2.4) shows that $tv_1 + (1-t)v_2 \in D_{k+1}$. This proves that D_{k+1} is closed and convex. From

$$\phi(v, z_k) \leq \phi(v, y_k) \iff 2\langle v, Jy_k - Jz_k\rangle \leq \|y_k\|^2 - \|z_k\|^2,$$

we also have that $E_{k+1} := \{v \in C_k : \phi(v, z_k) \le \phi(v, y_k)\}$ is closed and convex. In a similar way, we can prove that $F_{k+1} := \{v \in C_k : \phi(v, y_k) \le \phi(v, x_k)\}$ is closed and convex. Noticing that

$$C_{k+1} = D_{k+1} \cap E_{k+1} \cap F_{k+1},$$

we see that C_{k+1} is closed and convex. Then, for all $n \ge 1$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_0$ is well defined. Notice that $u_n = T_{r_n} z_n$ for all $n \ge 1$. From Lemma 6, one has that T_{r_n} is quasi-ϕ-nonexpansive mapping for each $n \ge 1$.

Next, we prove that $F \subset C_n$ for all $n \ge 1$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Then, for $\forall w \in F \subset C_k$, from Lemma 8, Lemma 10 and the condition $0 < \lambda_n \le c^2\alpha/2$ for all $n \ge 1$, one has

$$\begin{aligned}
&\phi(w, u_k) = \phi(w, T_{r_k} z_k) \le \phi(w, z_k) = \phi(w, Ty_k) \le \phi(w, \Pi_C[J^{-1}(Jx_k - \lambda_k Ax_k)]) \\
&\le \phi(w, J^{-1}(Jx_k - \lambda_k Ax_k)) = V(w, Jx_k - \lambda_k Ax_k) \\
&\le V(w, Jx_k - \lambda_k Ax_k + \lambda_k Ax_k) - 2\langle J^{-1}(Jx_k - \lambda_k Ax_k) - w, \lambda_k Ax_k\rangle \\
&\le \phi(w, x_k) - 2\lambda_k \langle J^{-1}(Jx_k - \lambda_k Ax_k) - x_k, Ax_k\rangle \\
&\qquad - 2\lambda_k\langle x_k - w, Ax_k - Aw\rangle - 2\lambda_k\langle x_k - w, Aw\rangle \\
&\le \phi(w, x_k) - 2\lambda_k\langle J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}Jx_k, Ax_k\rangle - 2\lambda_k\alpha\|Ax_k - Aw\|^2 \\
&\le \phi(w, x_k) + 2\lambda_k\|J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}Jx_k\|\|Ax_k\| - 2\lambda_k\alpha\|Ax_k - Aw\|^2 \\
&\le \phi(w, x_k) + \frac{4}{c^2}\lambda_k^2\|Ax_k - Aw\|^2 - 2\lambda_k\alpha\|Ax_k - Aw\|^2 \le \phi(w, x_k),
\end{aligned} \tag{2.5}$$

which shows that $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \ge 1$. From $x_n = \Pi_{C_n} x_0$, one sees

$$\langle x_n - z, Jx_0 - Jx_n\rangle \ge 0, \quad \forall z \in C_n. \tag{2.6}$$

Since $F \subset C_n$ for all $n \ge 1$, we arrive at

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall w \in F. \tag{2.7}$$

From Lemma 3, one has

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0),$$

for each $w \in F \subset C_n$ and for all $n \ge 1$. Therefore, the sequence $\phi(x_n, x_0)$ is bounded.

On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \ge 1.$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned}
\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) &\le \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned} \tag{2.8}$$

Letting $m, n \to \infty$ in (2.8), one has $\phi(x_m, x_n) \to 0$. It follows from Lemma 1 that $x_m - x_n \to 0$ as $m, n \to \infty$ Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that

$$x_n \to p \in C, \ (n \to \infty). \tag{2.9}$$

Next, we show that $p \in F(T)$. By taking $m = n+1$ in (2.8), one arrives at

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.10}$$

From Lemma 1, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.11}$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, z_n) \le \phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n).$$

It follows from (2.10) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = \lim_{n\to\infty} \phi(x_{n+1}, z_n) = \lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 1, one has

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.12}$$

Notice that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

It follows from (2.11) and (2.12) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.13}$$

It follows from (2.9) that

$$y_n \to p, \quad as\ n \to \infty. \tag{2.14}$$

On the other hand, one has

$$\|Ty_n - y_n\| \le \|Ty_n - x_{n+1}\| + \|x_{n+1} - y_n\| = \|z_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

From (2.12), one arrives at

$$\lim_{n\to\infty} \|Ty_n - y_n\| = 0$$

and it is easy to get $p \in F(T)$ from the closedness of T.

Next, we show that $p \in VI(C, A)$. Let Q be the maximal monotone operator defined by Lemma 9:

$$Qx := \begin{cases} Ax + N_C x, & if\ x \in C, \\ \emptyset, & if\ x \notin C. \end{cases}$$

For any given $(x, x') \in G(Q)$, we see that $x' - Ax \in N_C x$. Since $y_n \in C$, by the definition of $N_C x$, we have

$$\langle x - y_n, x' - Ax \rangle \geq 0.$$

On the other hand, from $y_n = \Pi_C[J^{-1}(Jx_n - \lambda_n Ax_n)]$ and Lemma 2, we obtain that

$$\langle x - y_n, Jy_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

from which it follows that

$$\langle x - y_n, \frac{Jy_n - Jx_n}{\lambda_n} + Ax_n \rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
&\langle x - y_n, x' \rangle \\
&\geq \langle x - y_n, Ax \rangle \\
&\geq \langle x - y_n, Ax \rangle - \langle x - y_n, \frac{Jy_n - Jx_n}{\lambda_n} + Ax_n \rangle \\
&= \langle x - y_n, Ax - Ay_n \rangle + \langle x - y_n, Ay_n - Ax_n \rangle - \langle x - y_n, \frac{Jy_n - Jx_n}{\lambda_n} \rangle \\
&\geq \langle x - y_n, Ay_n - Ax_n \rangle - \langle x - y_n, \frac{Jy_n - Jx_n}{\lambda_n} \rangle.
\end{aligned} \tag{2.15}$$

Since A is α-inverse strongly monotone, we have

$$\|Ay_n - Ax_n\| \leq \frac{1}{\alpha}\|y_n - x_n\|.$$

Thanks to (2.13), we obtain

$$\lim_{n\to\infty} \|Ay_n - Ax_n\| = 0.$$

From (2.15), we arrive at $\langle x - p, x' \rangle \geq 0$. Since A is maximal monotone, we obtain that $p \in A^{-1}(0)$ and hence $p \in VI(C, A)$.

Next, we show $p \in EF(f)$. Notice that

$$\|u_n - x_n\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

It follows from (2.11) and (2.12) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{2.16}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one obtains

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{2.17}$$

Noticing $u_n = T_{r_n} z_n$ and Lemma 7, for each $w \in F$, one has

$$\begin{aligned}
\phi(u_n, z_n) = \phi(T_{r_n} z_n, z_n) &\leq \phi(w, z_n) - \phi(w, T_{r_n} z_n) \\
&\leq \phi(w, x_n) - \phi(w, u_n) = 2\langle w, Ju_n - Jx_n \rangle + \|x_n\|^2 - \|u_n\|^2 \\
&\leq 2\|w\|\|Ju_n - Jx_n\| + (\|x_n\| + \|u_n\|)\|x_n - u_n\|.
\end{aligned}$$

From (2.16), (2.17) and Lemma 1, one arrives at

$$\|u_n - z_n\| \to 0, \ (n \to \infty). \tag{2.18}$$

From the assumption $r_n \geq a$, one sees

$$\lim_{n\to\infty} \frac{\|Ju_n - Jz_n\|}{r_n} = 0. \tag{2.19}$$

Since $u_n = T_{r_n} z_n$, one obtains

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C.$$

From the (A2), one arrives at

$$\|y - u_n\| \frac{\|Ju_n - Jz_n\|}{r_n} \geq \langle y - u_n, \frac{Ju_n - Jz_n}{r_n}\rangle \geq -f(u_n, y) \geq f(y, u_n), \ \forall y \in C.$$

By taking the limit as $n \to \infty$ in above inequality and from (A4) and (2.19), one has

$$f(y, p) \leq 0, \quad \forall y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from (A1) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

That is, $f(y_t, y) \geq 0$. Let $t \downarrow 0$, from (A3), we obtain $f(p, y) \geq 0$, for $\forall y \in C$. This implies that $p \in EP(f)$. This shows that $p \in F$.

Finally, we prove that $p = \Pi_F x_0$. By taking limit in (2.7), one has

$$\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in F.$$

At this point, in view of Lemma 2, one sees that $p = \Pi_F x_0$. This completes the proof. □

Remark 4. The highlight of Theorem 4 is as follows:

(a) We consider a more general nonlinear mapping, i.e. quasi-ϕ-nonexpansive mapping. To be more precise, we relax the strong restrictions $F(T) = \widetilde{F(T)}$ and $F(T) = \overline{F(T)}$ on the mapping T; see [5, 6, 14, 30] for more details;

(b) We remove the redundant iterative step "Y_n" in [13] and "W_n" in [30];

(c) We consider three hot problems of fixed point problems, variational inequality problems and equilibrium problems. In [30], the authors only studied fixed point problems and variational inequality problems. In [13], the authors only consider variational inequality problems.

As some applications of Theorem 4, we have the following results.

Corollary 1. Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let A an α-inverse-strongly monotone mapping from C into E^* and $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Assume that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = \Pi_C [J^{-1}(Jx_n - \lambda_n Ax_n)], \quad z_n = Ty_n, \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the duality mapping on E. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$ and $0 < \lambda_n \leq \frac{c^2\alpha}{2}$, for all $n \geq 1$, where c is the constant defined by (1.6). Then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap VI(C,A)} x_0$.

Remark 5. Corollary , which includes the results of Iiduka and Takahashi [13] as a special case, mainly improves and extends the corresponding result announced by Zegeye and Shahzad [30] in the following sense:

(1) Improve the mapping T from the relatively weak nonexpansive mapping to the quasi-ϕ-nonexpansive mapping. To be more precise, we relax the strong restriction $\overline{F(T)} = F(T)$.

(2) From computation point of view, we remove the iterative step "W_n"; see [30] for more details.

In the framework of Hilbert spaces, the following result can be deduced from Corollary 5 immediately.

Corollary 2. Let C be a nonempty and closed convex subset of a real Hilbert space H. Let A an α-inverse-strongly monotone mapping from C into H. Assume that $VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by:

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ C_{n+1} = \{v \in C_n : \|v - y_n\| \leq \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection from C onto C_{n+1}. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$ and $0 < \lambda_n \leq \alpha$, for all $n \geq 1$. Then $\{x_n\}$ converges strongly to $P_{VI(C,A)} x_0$.

Remark 6. Corollary 2 is an improvement of the corresponding results announced by Iiduka et al. [12]. We remove the redundant iterative "Q_n" in their iterative algorithms.

Corollary 3. Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$–$(A4)$, A an α-inverse-strongly monotone mapping from C into E^*. Assume that $EP(f) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = \Pi_C [J^{-1}(Jx_n - \lambda_n A x_n)], \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the duality mapping on E and $\{r_n\}$ is a positive number sequence such that $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C,A)$ and $0 < \lambda_n \leq \frac{c^2\alpha}{2}$ for all $n \geq 1$, where c is the constant defined by (1.6). Then $\{x_n\}$ converges strongly to $\Pi_{EP(f)\cap VI(C,A)} x_0$.

Proof. By taking $T = I$, the identity mapping, in Theorem 4, we can conclude the desired conclusion easily. □

In Hilbert spaces, Corollary 3 reduces to the following result.

Corollary 4. Let C be a nonempty and closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$–$(A4)$, A an α-inverse-strongly monotone mapping from C into H. Assume that $EP(f) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = P_C (x_n - \lambda_n A x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \|v - u_n\| \leq \|v - y_n\| \leq \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\{r_n\}$ is a positive number sequence such that $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C,A)$ and $0 < \lambda_n \leq \alpha$, for all $n \geq 1$. Then $\{x_n\}$ converges strongly to $P_{EP(f)\cap VI(C,A)} x_0$.

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References

[1] Ya.I. Alber. Metric and generalized projection operators in Banach spaces: properties and applications. In A.G. Kartsatos(Ed.), *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, pp. 15–50, New York, 1996. Marcel Dekker.

[2] Ya.I. Alber and S. Reich. An iterative method for solving a class of nonlinear operator equations in banach spaces. *PanAmer. Math. J.*, **4**:39–54, 1994.

[3] B. Beauzamy. *Introduction to Banach spaces and their Geometry, 2nd ed.* North Holland, 1995.

[4] E. Blum and W. Oettli. From optimization and variational inequalities to equilibrium problems. *Math. Student*, **63**:123–145, 1994.

[5] D. Butnariu, S. Reich and A.J. Zaslavski. Asymptotic behavior of relatively nonexpansive operators in Banach spaces. *J. Appl. Anal.*, **7**:151–174, 2001.

[6] D. Butnariu, S. Reich and A.J. Zaslavski. Weak convergence of orbits of nonlinear operators in reflexive Banach spaces. *Numer. Funct. Anal. Optim.*, **24**:489–508, 2003.

[7] L.C. Ceng and J.C. Yao. A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. Appl. Math.*, **214**:186–201, 2008.

[8] Y. Censor and S. Reich. Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization. *Optim.*, **37**:323–339, 1996.

[9] I. Cioranescu. *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*. Kluwer, Dordrecht, 1990.

[10] P.L. Combettes and S.A. Hirstoaga. Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.*, **6**:117–136, 2005.

[11] H. Iiduka. Hybrid conjugate gradient method for a convex optimization problem over the fixed-point set of a nonexpansive mapping. *J. Opti. Theory Appl.* (doi:10.1007/s10957-008-9463-6)

[12] H. Iiduka, W. Takahashi and M. Toyoda. Approximation of solutions of variational inequalities for monotone mappings. *PanAmer. Math. J.*, **14**:49–61, 2004.

[13] H. Iiduka and W. Takhashi. Strong convergence studied by a hybrid type method for monotone operators in a Banach space. *Nonlinear Anal.*, **68**:3679–368, 2008.

[14] S. Kamimura and W. Takahashi. Strong convergence of a proximal-type algorithm in a Banach space. *SIAM J. Optim.*, **13**:938–945, 2002.

[15] S.Y. Matsushita and W. Takahashi. A strong convergence theorem for relatively nonexpansive mappings in a Banach space. *J. Approx. Theory*, **134**:257–266, 2005.

[16] A. Moudafi. Second-order differential proximal methods for equilibrium problems. *J. Inequal. Pure Appl. Math.*, **4**, 2003. (art.18)

[17] A. Moudafi and P.-E.Maingé. Towards viscosity approximations of hierarchical fixed-point problems. *Fixed Point Theory Appl.*, **2006**, 2006. (Art. ID 95453)

[18] S. Ohsawa and W. Takahashi. Strong convergence theorems for resolvent of maximal monotone operators in Banach spaces. *Arch. Math.*, **81**:439–445, 2003.

[19] X. Qin, Y.J. Cho and S.M. Kang. Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. *J. Comput. Appl. Math.*, 2008. (doi:10.1016/j.cam.2008.06.011)

[20] X. Qin and Y. Su. Strong convergence theorems for relatively nonexpansive mappings in a Banach space. *Nonlinear Anal.*, **67**:1958–1965, 2007.

[21] S. Reich. A weak convergence theorem for the alternating method with Bregman distance. In A.G. Kartsatos(Ed.), *Theory and Applicationsof Nonlinear Operators of Accretive and Monotone Type*, pp. 313–318. Marcel Dekker, New York, 1996.

[22] R.T. Rockafellar. Monotone operators and proximal point algorithm. *SIAM J. Control Optim.*, **14**:877–898, 1976.

[23] M.V. Solodov and B.F. Svaiter. Forcing strong convergence of proximal point iterations in a Hilbert space. *Math. Program. Ser. A.*, **87**:189–202, 2000.

[24] Y. Su and X. Qin. Monotone CQ iteration processes for nonexpansive semigroups and maximal monotone operators. *Nonlinear Anal.*, **68**:3657–3664, 2008.

[25] A. Tada and W. Takahashi. Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem. *J. Optim. Theory Appl.*, **133**:359–370, 2007.

[26] S. Takahashi and W. Takahashi. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.*, **331**:506–515, 2007.

[27] W. Takahashi. *Nonlinear Functional Analysis*. Yokohama-Publishers, 2000.

[28] W. Takahashi and K. Zembayashi. Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. *Nonlinear Anal.*, **70**:45–57, 2009.

[29] H.K. Xu. Inequalities in Banach spaces with applications. *Nonlinear Anal.*, **16**:1127–1138, 1991.

[30] H. Zegeye and N. Shahzad. Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings. *Nonlinear Anal.*, 2008. (doi:10.1016/j.na.2008.03.058)