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Two-Point Boundary Value Problems at Resonance

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Abstract. Consider two-point boundary value problems of resonance type $x''+x=f(t,x,x'),\ x(0)=x(\pi)=0$. We investigate existence and multiplicity of solutions to such problems by using the quasilinearization process.

Key words: nonlinear boundary value problems, resonance, quasilinearization, multiple solutions.

1 Introduction

In this paper we consider the two-point nonlinear boundary value problem

$$x'' + x = f(t,\, x,\, x'), \tag{1.1}$$

$$x(0) = 0, \;\; x(\pi) = 0, \tag{1.2}$$

where $t \in I := [0, \pi],\ f \in C(I \times \mathbb{R}^2; \mathbb{R})$ (conditions on f will be imposed later). The problem under consideration is a problem at resonance because the linear part $\big(l_2 x\big)(t) := x'' + x$ is resonant with respect to the given boundary conditions (1.2). Indeed, the homogeneous problem

$$x'' + x = 0, \qquad x(0) = x(\pi) = 0$$

has nontrivial solutions and therefore the given problem (1.1), (1.2) may have no solutions at all, even if a continuous nonlinearity f is bounded.

On the other hand, it is well known that problem (1.1), (1.2) is solvable if

$$f = f(t) \quad \text{and} \quad \int_0^\pi f(t)\sin t\, dt = 0.$$

So one can try to find conditions on $f(t,\, x,\, x')$, which ensure the existence of a solution to the problem (1.1), (1.2).

The solvability of the problem under consideration has been studied by several authors. For instance, if $f = f(t,\, x) = h(t) - g(t,\, x)$, where $h \in L^2(I)$, $g \in C(I \times \mathbb{R}; \mathbb{R})$, then in accordance with the well-known Landesman-Lazer theorem [6] there exists at least one solution of the problem (1.1), (1.2) if the following inequality holds:

$$\int_0^\pi g_-(t) \sin t \, dt < \int_0^\pi h(t) \sin t \, dt < \int_0^\pi g_+(t) \sin t \, dt,$$

where $g_-(t) = \limsup_{x \to -\infty} g(t,\, x)$ and $g_+(t) = \liminf_{x \to \infty} g(t,\, x)$.

Some authors (see, for example, [5, 7]) consider that a key condition, for the existence of at least one solution of the problem (1.1), (1.2), is that the function f satisfies a monotonicity assumption with respect to the variable x. Other authors provide another solvability conditions for this problem [1, 2, 4].

We investigate solvability of the problem (1.1), (1.2) applying the quasilinearization method [8, 9]. We try to reduce the equation (1.1) to a quasi-linear one of the form

$$(L_2 x)\,(t) = F(t,\, x,\, x'), \tag{1.3}$$

where a function F is continuous, bounded and Lipschitzian in x and x' and the extracted linear part $(L_2 x)\,(t)$ is non-resonant with respect to the given boundary conditions (1.2), that means that the respective homogeneous problem

$$(L_2 x)\,(t) = 0, \quad x(0) = x(\pi) = 0$$

has only the trivial solution. If such a reduction is possible then in accordance with Conti's theorem [3] the modified problem (1.3), (1.2) is solvable. In addition, we use the fact, that an oscillatory type of a solution $x(t)$ of the quasi-linear problem (1.3), (1.2) corresponds to a type of non-resonance of the linear part in (1.3).

If a solution $x(t)$ of the modified quasi-linear problem (1.3), (1.2) is located in the domain $\Omega(t,\, x,\, x')$, where both equations (1.1) and (1.3) are equivalent, then the original problem at resonance (1.1), (1.2) has a solution of definite type.

If the equation (1.1) can be reduced to another quasi-linear equation

$$\big(L_2^* x\big)(t) = F^*(t,\, x,\, x'),$$

which is equivalent to (1.1) in different domain $\Omega^*(t,\, x,\, x')$ and moreover the linear parts $(L_2 x)\,(t)$ and $\big(L_2^* x\big)(t)$ are essentially different (i.e. with different types of non-resonance), then the original problem (1.1), (1.2) is expected to have multiple solutions.

2 Preliminaries

Consider a quasi-linear problem (1.3), (1.2). A linear part $(L_2 x)\,(t)$ in (1.3) for general case can be written in the form $x'' + p(t) x' + q(t) x$, where $p,\, q \in C(I; \mathbb{R})$. Several definitions will be used in the sequel.

Definition 1. A linear part $(L_2x)(t)$ in (1.3) is i-nonresonant with respect to the given boundary conditions (1.2) if a solution $x(t)$ of the Cauchy problem

$$(L_2x)(t) = 0,\ \ x(0) = 0,\ \ x'(0) = 1$$

has exactly i zeros in the interval $(0,\,\pi)$ and $x(\pi) \neq 0$.

If the linear parts $(L_2x)(t)$ and $(L_2^*x)(t)$ have different types of non-resonance we say for brevity that they are *essentially different*. For instance, the linear part $\big(x''+(1+3t)x\big)$ is 2-nonresonant with respect to the given boundary conditions (1.2) (see Fig. 1), but the linear part $\big(x''+(1+3\sin t)\,x\big)$ is 1-nonresonant (see Fig. 2), therefore these linear parts are essentially different.

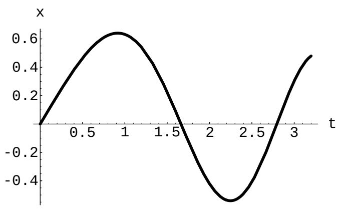


Figure 1. Solution of the problem $x''+(1+3t)\,x=0,\ x(0)=0,\ x'(0)=1$.

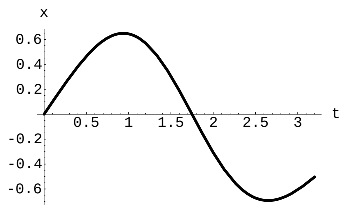


Figure 2. Solution of the problem $x''+(1+3\sin t)\,x=0,\ x(0)=0,\ x'(0)=1$.

Definition 2. Let $\xi(t)$ be a solution of the problem (1.3), (1.2) $\big($or (1.1), (1.2)$\big)$. We say that $\xi(t)$ is an i-type solution if there exists $\varepsilon > 0$ such that $\forall\,\delta \in (0,\varepsilon]$ the difference $u(t;\delta) = x(t;\delta) - \xi(t)$ has exactly i zeros in the interval $(0,\,\pi)$ and $u(\pi;\delta) \neq 0$, where $x(t;\delta)$ is a neighbouring solution for $\xi(t)$, i.e. it solves the same equation (1.3) $\big($or (1.1)$\big)$ and satisfies the initial conditions

$$\begin{cases} x(0;\delta) = \xi(0), \quad x'(0;\delta) = \xi'(0) + \delta\cdot\operatorname{sgn}\xi'(0), \quad \text{if} \quad \xi(t) \not\equiv 0, \\ x(0;\delta) = 0, \quad x'(0;\delta) = \delta \quad \text{if} \quad \xi(t) \equiv 0. \end{cases}$$

For instance, the trivial solution of the problem $x''+x = -7\arctan x$, (1.2) is 2-type solution because the difference between neighbouring solution and the trivial one in the interval $(0,\,\pi)$ has exactly two zeros (see Fig. 3), and the trivial solution of the problem $x''+x = -9\arctan x$, (1.2) is 3-type solution (see Fig. 4).

An i-type solution $\xi(t)$ of the problem (1.3), (1.2) $\big($or resp.: (1.1), (1.2)$)$ has the following characteristics in terms of the variational equation: a solution $y(t)$ of the variational equation

$$(L_2y)(t) = F_x(t,\,\xi(t),\,\xi'(t))y + F_{x'}(t,\,\xi(t),\,\xi'(t))y',$$

or resp.:

$$y''+y = f_x(t,\,\xi(t),\,\xi'(t))y + f_{x'}(t,\,\xi(t),\,\xi'(t))y'$$

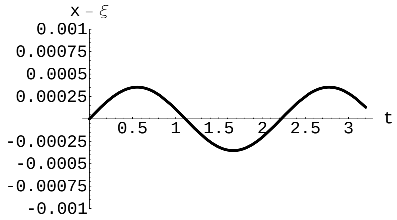


Figure 3. Difference between neighboring solution and trivial solution of the problem $x'' + x = -7 \arctan x$, $x(0) = 0,\ x(\pi) = 0$.

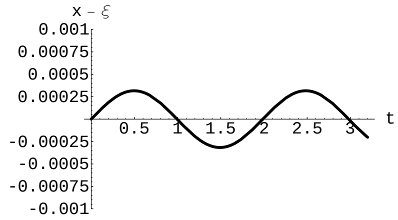


Figure 4. Difference between neighboring solution and trivial solution of the problem $x'' + x = -9 \arctan x$, $x(0) = 0,\ x(\pi) = 0$.

subject to the initial conditions

$$y(0) = 0, \quad y'(0) = 1$$

either has exactly i zeros in the interval $(0, \pi]$ or it has exactly i zeros in the interval $(0,\ \pi)$ and $y(\pi) = 0$. The cases of the i-th zero of $y(t)$ being at $t = \pi$ or $(i+1)$-th zero being at $t = \pi$ are not excluded.

The following result for quasi-linear problem (1.3), (1.2) was proved in [8, 9].

Theorem 1. *Quasi-linear problem* (1.3), (1.2) *with i-nonresonant linear part* $(L_2 x)(t)$ *has an i-type solution.*

Definition 3. Let equations (1.1) and (1.3), where the linear part $(L_2 x)(t)$ is non-resonant with respect to the boundary conditions under consideration, be equivalent in the domain

$$\Omega_N = \{(t, x, x') :\ 0 \le t \le \pi,\ |x| < N,\ |x'| < N_1\} \tag{2.1}$$

in the sense that any solution $x : I \to \mathbb{R}$ of (1.1) with a graph in Ω_N is also a solution of (1.3) and vice versa. Suppose that any solution $x(t)$ of the quasi-linear problem (1.3), (1.2) satisfies the estimates

$$|x(t)| < N, \quad |x'(t)| < N_1. \tag{2.2}$$

We will say then that the problem (1.1), (1.2) allows for quasilinearization with respect to a linear part $(L_2 x)(t)$.

Theorem 2. *If problem* (1.1), (1.2) *allows for quasilinearization with respect to some i-nonresonant linear part* $(L_2 x)(t)$*, then it has an i-type solution.*

Theorem 3. *Suppose that the problem* (1.1), (1.2) *allows for quasilinearization with respect to i-nonresonant linear part* $(L_2 x)(t)$ *in a domain* Ω_N *defined by* (2.1) *and, at the same time, it allows for quasilinearization with respect to j-nonresonant linear part* $(L_2^* x)(t)$ *in a domain*

$$\Omega_K = \{(t, x, x') :\ 0 \le t \le \pi,\ |x| < K,\ |x'| < K_1\},$$

where $i \ne j$. Then the problem (1.1), (1.2) *has at least 2 solutions of different types.*

Corollary 1. Suppose that the problem (1.1), (1.2) allows for quasilinearization with respect to n essentially different (in the sense of Definition 3) linear parts in n different domains of the form (2.1). Then it has at least n solutions of different types.

3 Quasilinearization of a Problem at Resonance

Quasilinearization process in some cases can be successfully applied to the boundary value problems of resonant type. If a problem at resonance allows for quasilinearization with respect to some non-resonant linear part then we can state an existence of a solution of definite type to this problem.

Quasilinearization process consists of the following stages:

1. Reduce equation (1.1) by extracting some non-resonant linear part;
2. Truncate the right hand side of the reduced equation;
3. Check the inequalities of the form (2.2).

3.1 Application 1

Consider the differential equation

$$x'' + x = -\lambda^2 \arctan x \tag{3.1}$$

together with the boundary condition (1.2). The equation (3.1) is equivalent to the following equation

$$x'' + (1 + k^2)x = k^2 x - \lambda^2 \arctan x.$$

If $k^2 \neq n^2 - 1$, $n \in \mathbb{N}$ then the extracted linear part $\big(x'' + (1 + k^2)x\big)$ is non-resonant with respect to (1.2). Let us denote $f_k(x) := k^2 x - \lambda^2 \arctan x$ and try to bound this function by a modulus of its local extremum

$$M_k = |f_k(x_{\text{ext}})| = \lambda^2 \arctan \sqrt{\frac{\lambda^2}{k^2} - 1} - k^2 \sqrt{\frac{\lambda^2}{k^2} - 1}. \tag{3.2}$$

Choose N_k such that

$$|x| \le N_k \quad \Rightarrow \quad |f_k(x)| \le M_k.$$

Such a number N_k can be calculated as a positive root of the equation

$$k^2 N - \lambda^2 \arctan N = \lambda^2 \arctan \sqrt{\frac{\lambda^2}{k^2} - 1} - k^2 \sqrt{\frac{\lambda^2}{k^2} - 1}. \tag{3.3}$$

Consider then instead of the function $f_k(x)$ the function

$$F_k(x) := f_k\big(\delta(-N_k, x, N_k)\big), \quad \text{where} \quad \delta(u, v, z) = \begin{cases} z, & v > z, \\ v, & u \le v \le z, \\ u, & v < u. \end{cases}$$

The modified quasi-linear equations (for different values of k)

$$x'' + (1 + k^2)x = F_k(x) \tag{3.4}$$

are equivalent to the given equation (3.1) in the respective domains

$$\Omega_k = \{(t, x) :\ 0 \le t \le \pi,\ |x| < N_k\}. \tag{3.5}$$

Theorem 4. *If there exists some* $k^2 \in (i^2 - 1,\, (i+1)^2 - 1)$, $i \in \mathbb{N}$, *which satisfies the inequality*

$$\frac{\pi}{\sqrt{1+k^2}\,|\sin(\pi\sqrt{1+k^2})|}\left|\lambda^2 \arctan\sqrt{\frac{\lambda^2}{k^2} - 1} - k^2\sqrt{\frac{\lambda^2}{k^2} - 1}\right| < N_k, \tag{3.6}$$

where N_k *is a positive root of the equation* (3.3)*, then there exists an* i*-type solution of the problem* (3.1),(1.2).

Proof. If $k^2 \in (i^2 - 1,\, (i+1)^2 - 1)$, $i \in \mathbb{N}$ then the extracted linear part in (3.4) is i-nonresonant with respect to (1.2). The modified problems (3.4),(1.2) are solvable and the respective solutions can be written in the integral form as

$$x_k(t) = \int_0^\pi G_k(t, s)\, F_k\big(x(s)\big)\, ds,$$

where $G_k(t,\, s)$ is the Green function of the respective homogeneous problem

$$x'' + (1 + k^2)x = 0, \quad x(0) = x(\pi) = 0.$$

The Green function satisfies the estimate

$$|G_k(t,\, s)| \le \Gamma_k := \frac{1}{\sqrt{1+k^2}\,|\sin(\pi\sqrt{1+k^2})|}. \tag{3.7}$$

Then we get that $|x_k(t)| \le \pi\Gamma_k M_k$. If a number k^2 is such that the inequality

$$\pi\Gamma_k M_k < N_k \tag{3.8}$$

holds, then the original problem (3.1),(1.2) allows for quasilinearization with respect to i-nonresonant linear part in the domain Ω_k defined by (3.5) and therefore it has an i-type solution. It follows from (3.7), (3.2), (3.3) that the inequality (3.8) reduces to (3.6). The proof is complete. □

We have computed results (see Table 1) for certain values of λ^2, which show that some numbers k^2 satisfy the inequality (3.6).

Example 1. In the boundary value problem

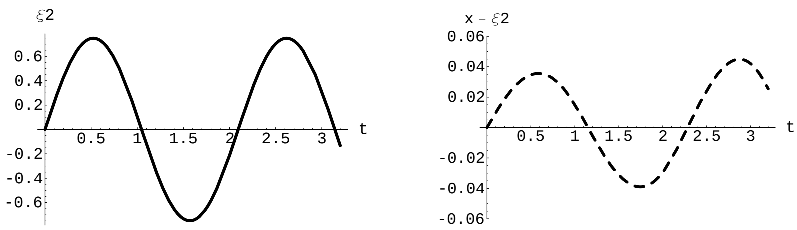
$$x'' + x = -9 \arctan x, \quad x(0) = x(\pi) = 0 \tag{3.9}$$

the parameter $\lambda^2 = 9$ and in accordance with results presented in Table 1 this problem allows for three essentially different quasilinearizations (for $k^2 = 1$,

Table 1. Results of calculations for the problem (3.1), (1.2).

λ^2	k^2	M_k	$\pi\,\Gamma_k$	$\pi\,\Gamma_k\,M_k$	N_k
$\lambda^2 = 7$	$k^2 = 1$	5.8329	2.3046	13.4427	16.4022
	$k^2 = 5$	0.7853	1.2987	1.0200	1.5568
$\lambda^2 = 8$	$k^2 = 1$	7.0297	2.3046	16.2008	19.1793
	$k^2 = 5$	1.3995	1.2987	1.8177	2.0745
$\lambda^2 = 9$	$k^2 = 1$	8.2502	2.3046	19.0137	21.9782
	$k^2 = 6$	1.2967	1.3238	1.7165	1.8180
	$k^2 = 8.5$	0.0799	3.9909	0.3189	0.5021

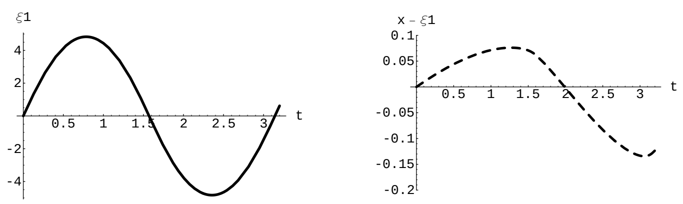
6 and 8.5). Then there exist at least 3 solutions of different types. We have computed all of them. First, it has the trivial solution, which is a 3-type solution (see Fig.4). Fig. 5 *a* illustrates the second solution given for the initial data $\xi_2(0) = 0$, $\xi_2'(0) = 2.283$. It is a 2-type solution because the difference between it and a neighbouring solution has exactly two zeros in the interval $(0, \pi)$ (see Fig. 5 *b* for $\delta = 0.1$).



a) b)

Figure 5. 2-type solution of the problem (3.9).

An 1-type solution of the problem (3.9) with the initial data $\xi_1(0) = 0$, $\xi_1'(0) = 10.651$ is presented in Fig. 6 *a*. Fig. 6 *b* illustrates the difference between this solution and its neighbouring solution (for $\delta = 0.1$).



a) b)

Figure 6. 1-type solution of the problem (3.9).

Example 2. The boundary value problem

$$x'' + x = -8 \arctan x, \quad x(0) = x(\pi) = 0, \tag{3.10}$$

in accordance with calculations presented in Table 1, allows two essentially different quasilinearizations. That means that there exist at least 2-type solution and 1-type solution of the problem (3.10). We have computed them, but the obtained solutions are different from the trivial solution. This fact shows that a problem at resonance can have a solution which can not be exposed applying the quasilinearization process. We will say for brevity that such a solution is *a resonant solution*.

A resonant solution $\eta(t)$ of the problem (1.1), (1.2) has the following characteristics in terms of the variational equation: a solution $y(t)$ of the variational equation

$$y'' + y = f_x(t,\, \eta(t),\, \eta'(t))y + f_{x'}(t,\, \eta(t),\, \eta'(t))y'$$

subject to the initial conditions $y(0) = 0$, $y'(0) = 1$ satisfies $y(\pi) = 0$. Indeed, if $\eta(t)$ is the trivial solution of the problem (3.10) ($\eta(t) \equiv 0$) then the respective variational equation is

$$y'' + y = -8y \tag{3.11}$$

and the function $y(t) = \frac{1}{3} \sin 3t$ is a solution of (3.11) which satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$ and $y(\pi) = 0$.

A resonant solution $\eta(t)$ of the problem (1.1), (1.2) has the following characteristics in terms of the neighbouring solutions: for small enough $\delta > 0$ the difference $u(t;\delta)$ between some neighbouring solution $x(t;\delta)$ and a resonant solution $\eta(t)$ satisfies the condition $u(\pi;\delta) = 0$.

3.2 Application 2

Consider the boundary value problem at resonance

$$x'' + x = h(t) - r(t,\, x)\, x \tag{3.12}$$

together with (1.2), where $t \in I := [0,\, \pi]$, $h \in C(I)$, $r, r_x \in C(I \times \mathbb{R}; \mathbb{R})$.

Suppose that the following conditions are satisfied:

(**A1**) r is even function in x;

(**A2**) $\lim\limits_{x\to\infty} r(t,\, x) = p(t), \quad p \in C\big(I;\, [0, +\infty)\big)$;

(**A3**) $\lim\limits_{x\to\infty} \Big(\dfrac{\partial r(t,\, x)}{\partial x}\, x^2\Big) = s(t), \quad s \in C(I)$.

The equation (3.12) is equivalent to equation

$$x'' + \big(1 + p(t)\big)x = h(t) - \big(r(t,\, x) - p(t)\big)x. \tag{3.13}$$

Theorem 5. *If a linear part* $\big(x'' + (1 + p(t))\, x\big)$ *is i-nonresonant with respect to the boundary conditions* (1.2) ($i \in \mathbb{N}$) *then there exists an i-type solution of the problem* (3.12), (1.2).

Proof. Denote $F(t,\, x) := h(t) - \big(r(t,\, x) - p(t)\big)x$. Conditions (**A1**)–(**A3**) imply that continuous function $F(t,\, x)$ is bounded in the domain

$$\Omega = \{(t,x):\ 0 \le t \le \pi,\ |x| < +\infty\}. \tag{3.14}$$

Thus if the extracted linear part $\big(x''+(1+p(t))\,x\big)$ is i-nonresonant with respect to (1.2) then the problem (3.13), (1.2) (and consequently the original problem (3.12), (1.2)) has an i-type solution. □

Example 3. Consider the problem

$$x'' + x = \cos t - t\left(3 - \frac{4}{\cosh x}\right)x, \quad x(0) = x(\pi) = 0. \tag{3.15}$$

It is a special case of the problem (3.12), (1.2), when $r(t,\, x) = t\Big(3 - \dfrac{4}{\cosh x}\Big)$, $h(t) = \cos t$ and conditions (**A1**)–(**A3**) are satisfied. Since $\lim\limits_{x\to\infty} r(t,\, x) = 3t$, then we can modify the equation in (3.15):

$$x'' + (1+3t)x = \frac{4\,t\,x}{\cosh x} + \cos t.$$

The right hand side function of the obtained equation is continuous and bounded in the domain (3.14). The extracted linear part $x'' + (1+3t)x$ is 2-nonresonant with respect to the boundary conditions (1.2) (see Fig. 1). Therefore there exists a 2-type solution of the problem (3.15), we have computed it.

Fig. 7 *a* illustrates the solution $\xi_2(t)$ of the problem (3.15) with initial data $\xi_2(0) = 0$, $\xi_2'(0) = 3.6$. This solution, actually, is a 2-type solution, because the difference between it and the neighbouring solution has exactly two zeros in the interval $(0,\, \pi)$ (see Fig. 7 *b* for $\delta = 0.1$).

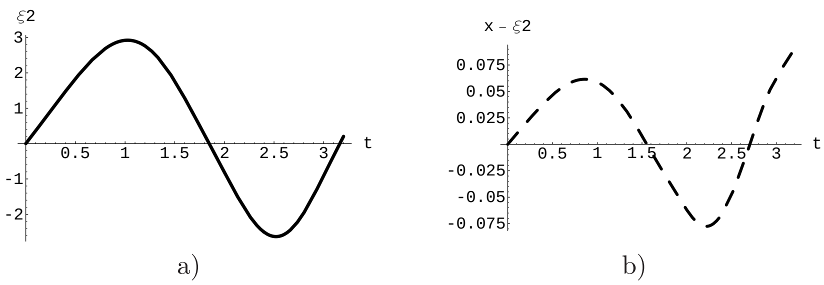


Figure 7. 2-type solution of the problem (3.15).

Continuously changing initial data we succeeded to compute the other solutions of problem (3.15). An 1-type solution $\xi_1(t)$ of (3.15) is depicted in Fig. 8 *a*, its initial data is given by $\xi_1(0) = 0$, $\xi_1'(0) = -0.11$. Fig. 8 *b* illustrates the difference between it and the neighbouring solution (for $\delta = -0.1$) and $\xi_1(t)$.

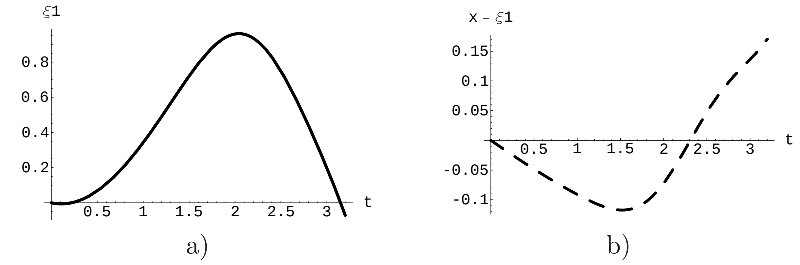


Figure 8. 1-type solution of the problem (3.15).

Fig. 9 *a* illustrates another solution $\xi_0(t)$ of the problem (3.15) subject to initial data $\xi_0(0) = 0$, $\xi_0'(0) = -0.739$. Here $\xi_0(t)$ is a 0-type solution, because the difference between neighboring solution and this one has no zeros in the interval $(0, \pi)$ (see Fig. 9 *b* for $\delta = -0.01$).

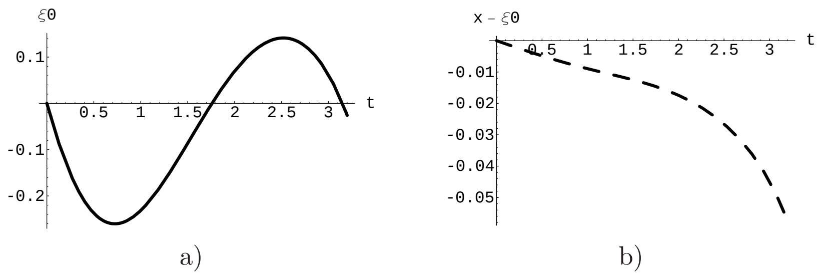


Figure 9. 0-type solution of the problem (3.15).

Notice that all computed solutions $\xi_2(t)$, $\xi_1(t)$ and $\xi_0(t)$ have exactly one zero in the interval $(0, \pi)$, but at the same time they are solutions of different types.

4 Conclusions

We see that in some cases a solvability of the boundary value problems at resonance can be proved by using the quasilinearization process. The respective cases are considered, when the differential equations of resonant type with bounded or unbounded right hand side function allow for quasilinearization. Multiplicity results for such boundary value problems are obtained. In the same time it is shown that a problem at resonance can have a resonant solution which can not be exposed applying the quasilinearization process.

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