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Similarity Solutions for Strong Shocks in a Non-Ideal Gas*

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Abstract. A group theoretic method is used to obtain an entire class of similarity solutions to the problem of shocks propagating through a non-ideal gas and to characterize analytically the state dependent form of the medium ahead for which the problem is invariant and admits similarity solutions. Different cases of possible solutions, known in the literature, with a power law, exponential or logarithmic shock paths are recovered as special cases depending on the arbitrary constants occurring in the expression for the generators of the transformation. Particular case of collapse of imploding cylindrically and spherically symmetric shock in a medium in which initial density obeys power law is worked out in detail. Numerical calculations have been performed to obtain the similarity exponents and the profiles of the flow variables behind the shock, and comparison is made with the known results.

Keywords: non-ideal gas, Lie group, similarity solutions, imploding shock.

AMS Subject Classification: 35L67;76L05.

1 Introduction

Many flow fields involving wave phenomena are governed by quasi linear hyperbolic system of partial differential equations (PDEs). For nonlinear systems involving discontinuities such as shocks, we do not generally have the complete exact solutions, and we have to rely on some approximate analytical or numerical methods which may be useful to provide information to understand the physics involved. One of the most powerful methods to obtain the similarity solutions to PDEs is similarity method which is based upon the study of their invariance with respect to one parameter Lie group of transformations. Indeed, with the help of infinitesimals and invariant surface conditions, one can construct similarity variables which can reduce these PDEs to ordinary differential equations (ODEs).

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A theoretical study of the imploding shock wave near the center of convergence, in an ideal gas was first performed by Guderley [7]. Among the extensive work that followed, we mention the contributions of Sakurai [17], Zeldovich and Raizer [23], Hayes [9], Ames [1], Axford and Holm [2, 3], Lazarus [11], Hafner [8], Sharma and Radha [19], Jena and Sharma [10], Conforto [6], Madhumita and Sharma [14], Sharma and Arora [18], Sharma and Radha [20] and Singh et al. [13] who presented high accuracy results and alternative approaches for the investigation of implosion problem. Steeb [21] determined the similarity solutions of the Euler equations and the Navier–Stokes equations for incompressible flows using the group theoretic approach outlined in the work of Bluman and Cole [4], Ovasiannikov [16], Olver [15], Logan [12] and Bluman and Kumei [5].

In the present paper, following Bluman and Kumei [5], and in a spirit closer to Logan [12], we characterize the medium ahead of shock for which the problem is invariant and admits similarity solutions. The occurrence of arbitrary constants in the expressions for the infinitesimals of the Lie group of transformations yields different cases of solutions with a power law, exponential or logarithmic shock paths. We have worked out in detail one particular case of collapse of imploding cylindrically and spherically symmetric shock in a medium in which initial density obeys power law.

We have compared our results with those obtained by Madhumita and Sharma [14], who employed a different technique to provide a solution to the implosion problem. Madhumita and Sharma [14] studied converging shock waves in a non-ideal medium. They have applied kinematics of one-dimensional motion to construct an evolution equation for strong cylindrical and spherical shock waves propagating into a gas at rest, and derived an infinite hierarchy of transport equations which describe the evolutionary behaviour of strong shocks propagating through an unsteady flow of a non-ideal gas.

The computed values of the similarity exponent are also compared with those obtained by using the Whitham's rule [22]; computation of the flow field in the region behind the shock has been carried out to determine the effects of parameter α and shock strength β.

2 Basic Equations and Shock Conditions

The basic equations describing the one-dimensional unsteady flow of a non-ideal gas, can be written as

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x + m\rho u/x &= 0, \\ \rho(u_t + uu_x) + p_x &= 0, \\ (x^m E)_t + (x^m u(E+p))_x &= 0, \end{aligned} \tag{2.1}$$

where ρ is the gas density, u the gas velocity, p the pressure, and $E = \rho e + \frac{1}{2}(\rho u^2)$ is the total energy density with e being the internal energy density; the independent variables are the space coordinate x and time t; and $m = 0, 1$ and 2 correspond to planar, cylindrical and spherical symmetry, respectively; the non-numeric subscripts denote the partial differentiation with respect to the indicated variables unless stated otherwise.

The equation of state, for motion in a non-ideal gas, is of the form:

$$p = \rho RT(1 + b\rho),$$

where b is the internal volume of the gas molecules which is known in terms of the molecular interaction potential; in high temperature gases, it is a constant with $b\rho \ll 1$. The gas constant R and the temperature T are assumed to obey the thermodynamic relations $R = C_p - C_v$, and $e = C_v T$, where $C_v = R/(\gamma - 1)$ is the specific heat at constant volume, C_p is the specific heat at constant pressure and γ is the ratio $C_p : C_v$. Thus in view of these thermodynamic relations, the equation of state can be written as:

$$p = \rho e\left(\gamma - 1 + \alpha\frac{\rho}{\rho_0}\right), \tag{2.2}$$

where $\alpha = (\gamma - 1)b\rho_0$ is the small parameter with ρ_0 being the density of the medium ahead of the shock. It may be noticed that the equation of state (2.2), characterizing the medium, is of Mie–Gruneisen type $p = \rho e\Gamma(\frac{\rho}{\rho_0})$, with $\Gamma(\frac{\rho}{\rho_0}) = (\gamma - 1 + \alpha\frac{\rho}{\rho_0})$ as the Gruneisen coefficient.

It follows immediately from equations $(2.1)_1$, $(2.1)_2$, (2.2) and the definition of E, that the equation $(2.1)_3$ may also be written as:

$$p_t + up_x + \frac{\gamma p}{\gamma - 1}\Gamma\left(\frac{\rho}{\rho_0}\right)(u_x + mu/x) = 0. \tag{2.3}$$

Let the initial condition at time $t = 0$ is given by $u = 0$, $\rho = \rho_0(x)$ and $p = p_0$, where the initial gas density $\rho_0(x)$ is a function of x and $p_0 > 0$ is an appropriate constant.

Now we consider the motion of a strong shock front propagating into the quiescent non-ideal gas of density ρ_0 considered above. The Rankine-Hugoniot jump conditions for the strong shock, $x = \varphi(t)$, give conditions just behind the shock as (see [22])

$$u = (1 - \beta^{-1})V, \quad \rho = \beta\rho_0, \quad p = \rho_0(1 - \beta^{-1})V^2, \tag{2.4}$$

where $V = \dot{\varphi}(t)$ is the shock speed, and β, which is a measure of the shock strength, is given by

$$\beta = \frac{\gamma + 1}{\gamma - 1}\left(1 - \frac{2\alpha}{(\gamma - 1)^2}\right). \tag{2.5}$$

3 Invariance under the Lie Group of Transformations

In order to obtain the similarity solutions of the system (2.1) we derive its symmetry group such that (2.1) is invariant under this group of transformations. The idea of the calculation is to find a one-parameter infinitesimal group of transformations (see [4, 5, 12]),

$$\begin{aligned} &x^* = x + \epsilon X(x, t, \rho, u, p), \quad t^* = t + \epsilon T(x, t, \rho, u, p),\\ &u^* = u + \epsilon\, U(x, t, \rho, u, p), \quad \rho^* = \rho + \epsilon S(x, t, \rho, u, p),\\ &p^* = p + \epsilon P(x, t, \rho, u, p), \end{aligned} \tag{3.1}$$

where X, T, U, S and P are the infinitesimals which are to be determined in such a way that the system (2.1), together with the jump conditions (2.4), is invariant under the group of transformations (3.1); the entity ϵ is a small parameter such that its square and higher powers may be neglected. The existence of such a group reduces the number of independent variables by one, which allows us to replace the system (2.1) of partial differential equations by a system of ordinary differential equations.

We introduce the notation $x_1 = x$, $x_2 = t$, $u_1 = u$, $u_2 = \rho$, $u_3 = p$ and $p^i_j = \partial u_i/\partial x_j$, where $i = 1, 2, 3$ and $j = 1, 2$. The system (2.1), which can be represented as

$$F_k(x_j, u_i, p^i_j) = 0, \quad k = 1, 2, 3,$$

is said to be constantly conformally invariant under the infinitesimal group of transformations (3.1) if and only if

$$LF_k = \alpha_{kr} F_r \quad \text{when } F_k = 0,\ r = 1, 2, 3, \tag{3.2}$$

where L is the extended infinitesimal generator of the group of transformations (3.1), and is given by

$$L = \xi^j \frac{\partial}{\partial x_j} + \eta^i \frac{\partial}{\partial u_i} + \beta^i_j \frac{\partial}{\partial p^i_j} \tag{3.3}$$

with $\xi^1 = X$, $\xi^2 = T$, $\eta^1 = U$, $\eta^2 = S$, $\eta^3 = P$, and

$$\beta^i_j = \frac{\partial \eta^i}{\partial x_j} + \frac{\partial \eta^i}{\partial u_k} p^k_j - \frac{\partial \xi^l}{\partial x_j} p^i_l - \frac{\partial \xi^l}{\partial u_n} p^i_l p^n_j, \tag{3.4}$$

where $l = 1, 2$, $n = 1, 2, 3$, $j = 1, 2$, $i = 1, 2, 3$, and $k = 1, 2, 3$; here repeated indices imply summation convention.

Equation (3.2) implies

$$\frac{\partial F_k}{\partial x_j} \xi^j + \frac{\partial F_k}{\partial u_i} \eta^i + \frac{\partial F_k}{\partial p^i_j} \beta^i_j = \alpha_{kr} F_r, \quad \text{when } F_k = 0,\ k = 1, 2, 3. \tag{3.5}$$

Substitution of β^i_j from (3.4) into (3.5) yields an identity in p^k_j and $p^i_l p^n_j$; hence we equate to zero the coefficients of p^i_j and $p^i_l p^n_j$ to obtain a system of first-order linear partial differential equations in the infinitesimals X, T, U, S and P. This system, called the system of determining equations of the group of transformations, is solved to find the invariance group of transformations.

We apply the above procedure to the system (2.1). The invariance of the continuity equation $(2.1)_1$ yields

$$\begin{aligned}
&S + \rho U_u - \rho X_x + u S_u = \alpha_{11}\rho + \alpha_{12}\rho u + \alpha_{13} K p \Gamma\big(\rho/\rho_0\big),\\
&U + \rho U_\rho + u S_\rho - u X_x - X_t = \alpha_{11} u,
\end{aligned}$$

$$S_u - \rho T_x = \alpha_{12}\rho, \quad \rho U_p + uS_p = \alpha_{12} + \alpha_{13}u, \tag{3.6}$$
$$S_\rho - uT_x - T_t = \alpha_{11}, \quad S_p = \alpha_{13},$$
$$S_t + uS_x + \rho U_x + U\frac{m\rho}{x} - mX\frac{\rho u}{x^2} = \alpha_{11}m\frac{\rho u}{x} + \alpha_{13}Kp\Gamma\left(\frac{\rho}{\rho_0}\right)\frac{mu}{x} + S\frac{mu}{x},$$

where $K = \gamma/(\gamma - 1)$, where the non-numeric subscripts denote partial differentiation with respect to the indicated variables.

Similarly, the invariance of the momentum equation $(2.1)_2$ yields

$$\rho U + uS + \rho uU_u - \rho uX_x + P_u - \rho X_t = \alpha_{21}\rho + \alpha_{22}\rho u + \alpha_{23}Kp\Gamma(\rho/\rho_0),$$
$$P_\rho + \rho uU_\rho = \alpha_{21}u, \quad P_p + \rho uU_p - X_x = \alpha_{22} + \alpha_{23}u,$$
$$S - \rho uT_x + \rho U_u - \rho T_t = \alpha_{22}\rho, \tag{3.7}$$
$$\rho U_\rho = \alpha_{21}, \qquad \rho U_p - T_x = \alpha_{23},$$
$$P_x + \rho U_t + \rho uU_x = \alpha_{21}\frac{m\rho u}{x} + \alpha_{23}\frac{mu}{x}Kp\Gamma(\rho/\rho_0).$$

Finally, the invariance of the energy equation $(2.1)_3$ yields

$$uP_u + pSK\frac{\alpha}{\rho_0} + PK\Gamma\left(\frac{\rho}{\rho_0}\right) + pK\Gamma\left(\frac{\rho}{\rho_0}\right)U_u - Kp\Gamma\left(\frac{\rho}{\rho_0}\right)X_x$$
$$= \alpha_{31}\rho + \alpha_{32}\rho u + \alpha_{33}Kp\Gamma(\rho/\rho_0),$$
$$Kp\Gamma(\rho/\rho_0)U_\rho + uP_\rho = \alpha_{31}u,$$
$$U + Kp\Gamma(\rho/\rho_0)U_p + uP_p - uX_x - X_t = \alpha_{32} + \alpha_{33}u,$$
$$P_u - Kp\Gamma(\rho/\rho_0)T_x = \alpha_{32}\rho, \tag{3.8}$$
$$P_\rho = \alpha_{31}, \qquad P_p - uT_x - T_t = \alpha_{33},$$
$$\Big(P_t - XKp\Gamma(\rho/\rho_0)\frac{mu}{x^2} + SKp\frac{\alpha}{\rho_0}\frac{mu}{x} + PK\Gamma(\rho/\rho_0)\frac{mu}{x}$$
$$+ Kp\Gamma(\rho/\rho_0)U_x + uP_x + pUK\Gamma(\rho/\rho_0)\frac{m}{x}\Big)$$
$$= \alpha_{31}\frac{m\rho u}{x} + \alpha_{33}Kp\Gamma(\rho/\rho_0)\frac{mu}{x},$$

where $K = \gamma/(\gamma - 1)$. We solve the system of determining equations (3.6), (3.7) and (3.8) to obtain

$$X = \begin{cases}(\alpha_{22} + 2a)x + c_2t + c_3, & \text{if } \alpha \neq 0,\\ (\alpha_{22} - \alpha_{11} + a)x + c_2t + c_3, & \text{if } \alpha = 0,\end{cases} \; T = at + b,$$
$$S = \begin{cases}0, & \text{if } \alpha \neq 0,\\ (\alpha_{11} + a)\rho, & \text{if } \alpha = 0,\end{cases} \qquad U = \begin{cases}(\alpha_{22} + a)u + c_2, & \text{if } \alpha \neq 0,\\ (\alpha_{22} - \alpha_{11})u + c_2, & \text{if } \alpha = 0,\end{cases}$$
$$P = \begin{cases}2(\alpha_{22} + a)p, & \text{if } \alpha \neq 0,\\ (2\alpha_{22} - \alpha_{11} + a)p, & \text{if } \alpha = 0,\end{cases} \tag{3.9}$$

where α_{11}, α_{22}, a, b, c_2 and c_3 are arbitrary constants. It is also found that $\alpha_{11} = -a$ for $\alpha \neq 0$. Thus, the infinitesimals of the invariant group of transformations are completely known.

4 Similarity Solutions

The arbitrary constants, which appear in the expressions for the infinitesimals of the invariant group of transformations, lead to various cases of possible solutions as discussed below.

Case I. When $\alpha_{22} + 2a \neq 0$ and $a \neq 0$ for $\alpha \neq 0$, ($\alpha_{22} - \alpha_{11} + a \neq 0$ and $a \neq 0$ for $\alpha = 0$), using the translational invariance of x and t, we obtain from equation (3.9)

$$X = \begin{cases} (\alpha_{22} + 2a)x + c_2 t, & \text{if } \alpha \neq 0, \\ (\alpha_{22} - \alpha_{11} + a)x + c_2 t, & \text{if } \alpha = 0, \ T = at. \end{cases} \tag{4.1}$$

However, the infinitesimals U, S and P in set (3.9) remain unchanged. To obtain the similarity solutions, we use the invariant surface conditions (see [4] and [12]), which yield

$$X\rho_x + T\rho_t = S, \quad Xu_x + Tu_t = U, \quad Xp_x + Tp_t = P. \tag{4.2}$$

Using (3.9) and (4.1), we integrate (4.2) to obtain for planar flow:

$$\rho = \begin{cases} \hat{S}(\xi) & \text{if } \alpha \neq 0, \\ t^{(\alpha_{11}+a)/a}\hat{S}(\xi) & \text{if } \alpha = 0, \end{cases} \qquad u = \begin{cases} t^{(\delta-1)}\hat{U}(\xi) & \text{if } m = 1,2, \\ t^{(\delta-1)}\hat{U}(\xi) - k^* & \text{if } m = 0, \end{cases}$$

$$p = \begin{cases} t^{2(\delta-1)}\hat{P}(\xi) & \text{if } \alpha \neq 0, \\ t^{(2\delta-1+\alpha_{11}/a)}\hat{P}(\xi) & \text{if } \alpha = 0, \end{cases} \tag{4.3}$$

where $\hat{S}$, $\hat{U}$ and $\hat{P}$ are the functions of a similarity variable ξ, which is found as

$$\xi = \begin{cases} x/(At^\delta) & \text{if } m = 1,2, \\ (x + A^* t)(At^\delta) & \text{if } m = 0, \end{cases} \tag{4.4}$$

where A is a dimensional constant. Let $\xi = 1$ be the basic position of the shock; then the shock path $\varphi(t)$ and the shock velocity V are obtained as

$$\varphi = \begin{cases} At^\delta & \text{if } m = 1,2, \\ At(t^{\delta-1} - A^*/A) & \text{if } m = 0, \end{cases} \quad V = \begin{cases} \delta\varphi/t & \text{if } m = 1,2, \\ A\delta t^{\delta-1} - A^* & \text{if } m = 0. \end{cases} \tag{4.5}$$

The boundary conditions at the shock become

$$\rho|_{\xi=1} = \begin{cases} \hat{S}(1) & \text{if } \alpha \neq 0, \\ t^{(\alpha_{11}+a)/a}\hat{S}(1) & \text{if } \alpha = 0, \end{cases} \quad u|_{\xi=1} = \begin{cases} t^{(\delta-1)}\hat{U}(1) & \text{if } m = 1,2, \\ t^{(\delta-1)}\hat{U}(1) - k^* & \text{if } m = 0, \end{cases}$$

$$p|_{\xi=1} = \begin{cases} t^{2(\delta-1)}\hat{P}(1) & \text{if } \alpha \neq 0, \\ t^{(2\delta-1+\alpha_{11}/a)}\hat{P}(1) & \text{if } \alpha = 0. \end{cases}$$

Invariance of the jump conditions suggests that $\rho_0(x)$ must be of the following form

$$\rho_0 = \begin{cases} \rho_c & \text{if } \alpha \neq 0, \\ \rho_c(x/x_1)^\mu & \text{if } \alpha = 0, \end{cases} \tag{4.6}$$

where ρ_c and $x_1 \neq 0$ are dimensional constants and μ is a dimensionless constant.

Now applying the jump conditions (2.4) and using (4.5), we obtain

$$\hat{S}(1) = \begin{cases} \beta\rho_c & \text{if } \alpha \neq 0, \\ \beta\rho_c(A/x_1)^\mu & \text{if } \alpha = 0, \end{cases} \qquad \hat{U}(1) = A\delta(1 - 1/\beta),$$
$$\hat{P}(1) = \begin{cases} \rho_c A^2\delta^2(1 - 1/\beta) & \text{if } \alpha \neq 0, \\ \rho_c A^2\delta^2(A/x_1)^\mu(1 - 1/\beta) & \text{if } \alpha = 0, \end{cases} \tag{4.7}$$

where $\delta = (\alpha_{22} + 2a)/a$. Thus, substituting (4.5) and (4.6) into (4.3), we obtain the following functional form of ρ, u and p in the entire flow field:

$$\rho = \begin{cases} \rho_c S^*(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))S^*(\xi) & \text{if } \alpha = 0, \end{cases} \qquad p = \begin{cases} \rho_c V^2 P^*(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))V^2 P^*(\xi) & \text{if } \alpha = 0, \end{cases}$$
$$u = VU^*(\xi), \tag{4.8}$$

where

$$S^*(\xi) = \begin{cases} \hat{S}(\xi)/\rho_c & \text{if } \alpha \neq 0, \\ x_1^\mu \hat{S}(\xi)/(\rho_c A^\mu) & \text{if } \alpha = 0, \end{cases} \quad P^*(\xi) = \begin{cases} \hat{P}(\xi)/(\rho_c\delta^2 A^2) & \text{if } \alpha \neq 0, \\ x_1^\mu \hat{P}(\xi)/(\rho_c\delta^2 A^{\mu+2}) & \text{if } \alpha = 0. \end{cases}$$
$$U^*(\xi) = \hat{U}(\xi)/(A\delta),$$

Substitution of (4.8) into system (2.1) of governing equation and use of (4.6) yield the following system of ordinary differential equations in S, U and P:

$$(U - \xi)S' + S\Big(U' + \frac{mU}{\xi}\Big) = 0,$$
$$\frac{(\delta - 1)}{\delta}U + (U - \xi)U' + \frac{P'}{S} = 0, \tag{4.9}$$
$$2\frac{(\delta - 1)}{\delta}P + (U - \xi)P' + \Big(\gamma P + \frac{\alpha\gamma}{\gamma - 1}PS\Big)\Big(U' + \frac{mU}{\xi}\Big) = 0,$$

where $'$ denotes the differentiation with respect to the similarity variable ξ. Also the following conditions are obtained from the jump conditions (2.4),

$$S(1) = \beta, \qquad U(1) = 1 - \beta^{-1}, \qquad P(1) = 1 - \beta^{-1}. \tag{4.10}$$

In the next section the system (4.9) together with the initial conditions (4.10) is integrated numerically to obtain the flow field.

Case II. When $\alpha_{22} \neq 0$ and $a = 0$ for $\alpha \neq 0$ ($\alpha_{22} - \alpha_{11} \neq 0$ and $a = 0$ for $\alpha = 0$), using the translational invariance of x and t, we obtain the following forms of the similarity solutions

$$\rho = \begin{cases} \rho_c\ S(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))S(\xi) & \text{if } \alpha = 0, \end{cases} \qquad p = \begin{cases} \rho_c V^2\ P(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))V^2 P(\xi) & \text{if } \alpha = 0, \end{cases}$$
$$u = VU(\xi), \tag{4.11}$$

where S, U and P are the functions of a similarity variable ξ, which is found as $\xi = \frac{x}{x_1}e^{-\delta t/A}$, where x_1 and A are arbitrary dimensional constants and

$$\delta = \begin{cases} \alpha_{22}/b & \text{if } \alpha \neq 0, \\ (\alpha_{22} - \alpha_{11})/b & \text{if } \alpha = 0. \end{cases}$$

Let $\xi = 1$ be the basic position of the shock; then the shock path $x = \varphi(t)$ and the shock velocity V are found as

$$\varphi(t) = x_1 e^{\delta t/A}, \qquad V(t) = \frac{x_1 \delta}{A} e^{\delta t/A}. \tag{4.12}$$

It may be noticed that in this case the shock path is exponential and is given by $(4.12)_1$. Invariance of the jump conditions (2.4) suggests that $\rho_0(x)$ must be of the following form

$$\rho_0 = \begin{cases} \rho_c & \text{if } \alpha \neq 0, \\ \rho_c (x/x_1)^{\mu} & \text{if } \alpha = 0, \end{cases} \tag{4.13}$$

where ρ_c is arbitrary dimensional constant, and

$$\mu = \begin{cases} \alpha_{11}/\alpha_{22} & \text{if } \alpha \neq 0, \\ \alpha_{11}/(\alpha_{22} - \alpha_{11}) & \text{if } \alpha = 0. \end{cases} \tag{4.14}$$

Substitution of (4.11) into system (2.1) of governing equation and use of (4.13) yield the following system of ordinary differential equations in S, U and P for $\alpha \neq 0$:

$$\begin{aligned} (U - \xi)S' + S\Big(U' + \frac{mU}{\xi}\Big) &= 0, \\ U + (U - \xi)U' + \frac{P'}{S} &= 0, \\ 2P + (U - \xi)P' + \Big(\gamma P + \frac{\alpha\gamma}{\gamma - 1}PS\Big)\Big(U' + \frac{mU}{\xi}\Big) &= 0, \end{aligned} \tag{4.15}$$

where $'$ denotes the differentiation with respect to the similarity variable ξ. System (4.15), together with the initial conditions (4.10), may be integrated numerically to obtain the flow field.

Case III. Let us consider the case when $\alpha_{22} + 2a = 0$ and $a \neq 0$ for $\alpha \neq 0$ ($\alpha_{22} - \alpha_{11} + a = 0$ and $a \neq 0$ for $\alpha = 0$). In this case, there does not exist any similarity solution for the non-planar flows; however, for the planar flow similarity solution does exist. The following similarity solution for the present case is found from (3.9) and (4.2) for the planar flow

$$\rho = \begin{cases} \rho_c S(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))S(\xi) & \text{if } \alpha = 0, \end{cases} \qquad p = \begin{cases} \rho_c V^2 P(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))V^2 P(\xi) & \text{if } \alpha = 0, \end{cases}$$

$$u = VU(\xi), \tag{4.16}$$

where S, U and P are the functions of a similarity variable ξ, which is found as $\xi = (x - x_1\delta \ln(t/A))/x_1$, here x_1 and A are arbitrary dimensional constants and $\delta = c_3/a$.

Let $\xi = 0$ be the basic position of the shock; then the shock path $x = \varphi(t)$ and the shock velocity V are found as

$$\varphi(t) = x_1\delta\ln(t/A), \qquad V = \delta x_1/t. \tag{4.17}$$

Invariance of the jump conditions suggests that $\rho_0(x)$ must be of the following form

$$\rho_0 = \begin{cases} \rho_c & \text{if } \alpha \neq 0, \\ \rho_c e^{\mu x/x_1} & \text{if } \alpha = 0, \end{cases}$$

where ρ_c is a dimensional constant and $\mu = (\alpha_{11} + a)/c_3$. It may be noticed that in this case the shock path is logarithmic and is given by $(4.17)_1$.

Substitution of (4.16) into the system (2.1) yields the following system of ordinary differential equations in S, U and P:

$$\begin{aligned} (U-1)S' + SU' &= 0, \\ U'(U-1) - \frac{U}{\delta} + \frac{P'}{S} &= 0, \\ (U-1)P' - \frac{2}{\delta}P + \Big(\gamma P + \frac{\alpha\gamma}{\gamma-1}SP\Big)U' &= 0. \end{aligned} \tag{4.18}$$

Also the following conditions are obtained from the jump conditions (2.4),

$$S(0) = \beta, \qquad U(0) = 1 - \beta^{-1}, \quad P(0) = 1 - \beta^{-1}. \tag{4.19}$$

The system (4.18) together with the conditions (4.19) may be integrated numerically to obtain the flow field.

Case IV. Let us consider the case when $\alpha_{22} = 0$ and $a = 0$ for $\alpha \neq 0$ ($\alpha_{22} - \alpha_{11} = 0$ and $a = 0$ for $\alpha = 0$). Making use of the translational invariance of the system (2.1) with respect to x and t, we find from (3.9) and (4.2) the following self-similar solution:

$$\rho = \begin{cases} \rho_c S(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))S(\xi) & \text{if } \alpha = 0, \end{cases} \qquad p = \begin{cases} \rho_c V^2 P(\xi) & \text{if } \alpha \neq 0, \\ \rho_0(\varphi(t))V^2 P(\xi) & \text{if } \alpha = 0, \end{cases}$$
$$u = VU(\xi), \tag{4.20}$$

where dimensionless similarity variable ξ is

$$\xi = \frac{1}{x_1}\Big(x - x_1\frac{\delta\, t}{A}\Big),$$

and the shock position φ and velocity V are given by

$$\varphi = x_1(1 + \delta t/A), \qquad V = \delta x_1/A, \tag{4.21}$$

where x_1 and A are the dimensional constants, and $\delta = c_3/b$ is a constant. It may be noted that the shock path is linear in this case and is given by $(4.21)_1$.

The invariance of the jump conditions implies that density ahead of the shock must be of the following form

$$\rho_0 = \begin{cases} \rho_c & \text{if } \alpha \neq 0, \\ \rho_c \ e^{\mu x/x_1} & \text{if } \alpha = 0, \end{cases}$$

where ρ_c is an arbitrary dimensional constant and $\mu = \alpha_{11}/c_3$.

Substitution of (4.20) into the system (2.1) leads to the following system of ordinary differential equations in S, U and P:

$$\begin{aligned} (U-1)S' + SU' &= 0, \\ (U-1)U' + \frac{P'}{S} &= 0, \\ (U-1)P' + \left(\gamma P + \frac{\alpha\gamma}{\gamma-1} SP\right)U' &= 0, \end{aligned} \tag{4.22}$$

where $'$ denotes the differentiation with respect to the similarity variable ξ. The system (4.22) together with the conditions (4.10) may be integrated numerically to obtain the flow field.

5 Shock Implosion

Here, we consider the problem in Case I of an imploding shock for which $V \gg a_0$ in the neighbourhood of implosion. For the problem of a converging shock collapsing to the center/axis, the origin of time t is taken to be the instant at which the shock reaches the center/axis so that $t \leq 0$ in (4.9). In this regard the definition of the similarity variable is slightly modified by setting

$$\varphi(t) = A(-t)^{\delta}, \qquad \xi = \frac{x}{A(-t)^{\delta}}, \tag{5.1}$$

so that the intervals of the variables are: $-\infty < t \leq 0$, $0 \leq x \leq \varphi$ and $1 \leq \xi < \infty$.

As the gas density, velocity, pressure and the sound speed at any finite radius are bounded, at the instant of collapse $t = 0$, where $\xi = \infty$ for $x \neq 0$, it follows that for $\rho = \rho_c S(\xi)$, $u = (\delta\varphi/t)U(\xi)$, $p = \rho_c(\delta\varphi/t)^2 P(\xi)$ and $a^2 = \gamma(\delta\varphi/t)^2(P/S)$ to remain bounded there, we must have

$$U(\infty) = 0, \qquad P(\infty)/S(\infty) = 0. \tag{5.2}$$

Thus, equations (4.9), (4.10) and (5.2) together constitute a boundary value problem (BVP), solving which we can determine the flow field behind the shock.

The system (4.9) is written in the matrix form as $AW' = C$, where $W = (U, S, P)^{tr}$, and the matrix A and the column vector C can be read off by inspection of the equations (4.9); $'$ is the derivative with respect to ξ. It may be noted that the system (4.9) has an unknown parameter δ, which we compute by an iterative procedure. We solve the system (4.9) for the derivatives U', S' and P' as

$$U' = \Omega_1/\Omega, \quad S' = \Omega_2/\Omega, \quad P' = \Omega_3/\Omega, \tag{5.3}$$

where Ω, which is the determinant of the matrix A, is given by $\Omega = (\xi - U)[(U-\xi)^2 - \gamma\frac{P}{S}(1+\frac{\alpha S}{\gamma-1})]$, and Ω_j $(j = 1,2,3)$ are the determinants which we obtain from Ω by replacing the jth column by the column vector C.

It may be noted that $\Omega > 0$ at $\xi = 1$ and $\Omega < 0$ at $\xi = \infty$; hence there exists a $\xi_c \in [1,\infty)$ where Ω vanishes, thereby the solutions become singular at $\xi = \xi_c$. In order to obtain a nonsingular solution of the system (4.9) in the interval $[1,\infty)$, we choose the exponent δ such that Ω vanishes only at the points where the determinant Ω_1 is also zero; it can be checked that at points where Ω and Ω_1 vanish, the determinants Ω_2 and Ω_3 also vanish simultaneously. To find the exponent δ, we introduce the variable Z, as $Z = (U-\xi)^2 - \gamma\frac{P}{S}(1+\frac{\alpha S}{\gamma-1})$, which, in view of (5.3) implies

$$dZ/d\xi = \Omega_4/\Omega, \tag{5.4}$$

where

$$\Omega_4 = 2(U-\xi)(\Omega_1-\Omega) - \frac{\alpha\gamma P}{(\gamma-1)S}\Omega_2 + \frac{\gamma}{S^2}\left(1+\frac{\alpha S}{\gamma-1}\right)(P\Omega_2 - S\Omega_3). \tag{5.5}$$

Equations (5.3), in view of (5.4), become

$$\frac{dU}{dZ} = \frac{\Omega_1}{\Omega_4}, \qquad \frac{dS}{dZ} = \frac{\Omega_2}{\Omega_4}, \qquad \frac{dP}{dZ} = \frac{\Omega_3}{\Omega_4}, \tag{5.6}$$

where the initial conditions are given in (4.10). Also the variable ξ is obtained from the expression of Z, as $\xi = U + (Z + \gamma\frac{P}{S}(1+\frac{\alpha S}{\gamma-1}))^{1/2}$.

Table 1. Comparison of the similarity exponent δ obtained in the present method with that obtained by Whitham's rule [22] and Madhumita and Sharma [14] for different values of α and β with $m = 1$ and $\gamma = 7/5$.

α	β	Present δ	Whitham's rule [22] δ	Madhumita and Sharma [14] δ
0.00	6.00	0.83527	0.835373	0.835306
0.001	5.925	0.8330	0.834012	0.8338205
0.003	5.775	0.83048	0.831223	0.830894
0.005	5.625	0.82749	0.828339	0.828024
0.007	5.475	0.82474	0.825355	0.8251775
0.009	5.325	0.82219	0.822265	0.824143
0.01	5.25	0.82099	0.820678	0.822701
0.015	4.875	0.81564	0.812301	0.818906

For solving equations (5.6) and (4.10), we apply fourth order Runge–Kutta method. In order to compute the value of δ, we integrate the equations (5.6) from the shock, $Z = Z(1)$ to the singular point $Z = 0$ after choosing a trial value of δ, and evaluate U, S, P and Ω_1 at $Z = 0$. If Ω_1 vanishes at this point, then we have started with the correct value of δ; otherwise we change this initial guess of δ and repeat the same procedure again and again till the corrected value of δ yields a zero value of Ω_1 when $Z = 0$. For $\gamma = 7/5$ and different values of α and β, the values of the similarity exponent δ, obtained from the above procedure, are listed in Tables 1 and 2; also they are compared with the corresponding values obtained by CCW approximation [22] and Madhumita and Sharma [14]. Indeed, the computed values are in excellent agreement with those obtained by CCW approximation [22] and Madhumita and Sharma [14].

Table 2. Comparison of the similarity exponent δ obtained in the present method with that obtained by Whitham's rule [22] and Madhumita and Sharma [14] for different values of α and β with $m = 2$ and $\gamma = 7/5$.

α	β	Present δ	Whitham's rule [22] δ	Madhumita and Sharma [14] δ
0.00	6.00	0.71718	0.717287	0.717174
0.001	5.925	0.71441	0.715284	0.714644
0.003	5.775	0.70918	0.711191	0.709833
0.005	5.625	0.70434	0.706979	0.705248
0.007	5.475	0.69989	0.702642	0.700797
0.009	5.325	0.69576	0.698174	0.698878
0.01	5.25	0.68898	0.69589	0.696718
0.015	4.875	0.68560	0.683929	0.690714

The velocity, density and pressure profiles for the present case are shown in Figs. 1 to 6. Figs. 1, 2 and 3 correspond to cylindrical shocks and Figs. 4, 5 and 6 to spherical shocks. The CCW approximation consists in applying the differential relation, which is valid on characteristics moving in the same directions as the shock, to the flow quantity immediately behind the shock. The computed results are given in Tables 1 and 2 for $m = 1$ and $m = 2$, respectively. Tables 1 and 2 show that an increase in parameter α causes δ to decrease and consequently the shock velocity to increase as shock approaches the center/axis.

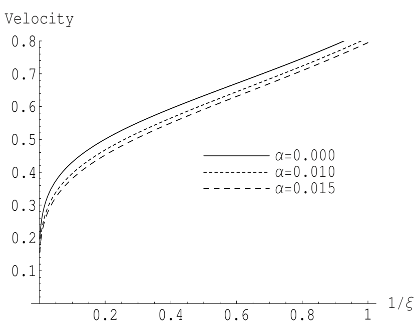

Figure 1. Velocity profiles for cylindrically symmetric ($m = 1$); $\gamma = 7/5$.

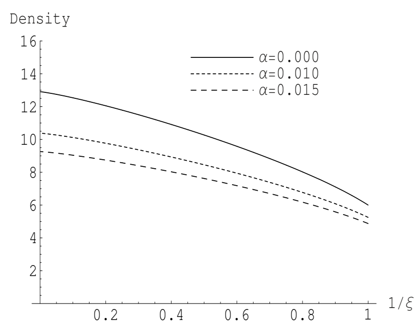

Figure 2. Density profiles for cylindrically symmetric ($m = 1$) flow; $\gamma = 7/5$.

We also obtain from (5.3) the analytical expressions of the flow variables at the instant of collapse of the shock at $t = 0$, $\varphi = 0$ ($t = 0$, $x \neq 0$ correspond to $\xi = \infty$), where conditions (5.2) hold; they are given as

$$U \sim \xi^{(\delta-1)/\delta}, \qquad S \sim \xi^{\mu}, \qquad P \sim \xi^{\mu+2(\delta-1)/\delta} \quad \text{as } \xi \to \infty. \tag{5.7}$$

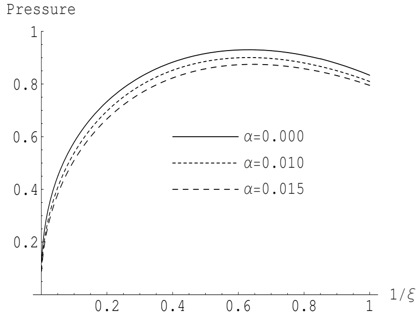

Figure 3. Pressure profiles for cylindrically symmetric ($m = 1$) flow; $\gamma = 7/5$.

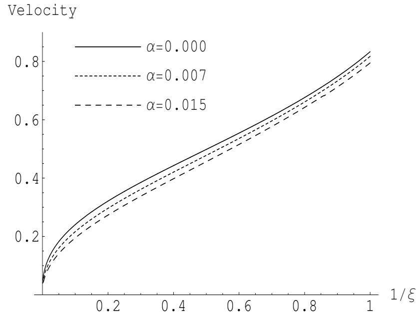

Figure 4. Velocity profiles for spherically symmetric ($m = 2$); $\gamma = 7/5$.

The first relation in (5.7) implies that the velocity U tends to zero at the instant of collapse where $\xi \to \infty$ since δ is always smaller than one. It may be noted from (5.7) that for uniform initial density (i.e. $\mu = 0$) the density S behind the shock remains bounded at the time of collapse while it becomes unbounded at the time of collapse for non-uniform density (i.e. $\mu > 0$). Equation (5.7) also indicates that the pressure P behind the shock remains bounded (unbounded) at the instant of collapse if $\mu + 2(\delta - 1)/\delta$ is negative (positive).

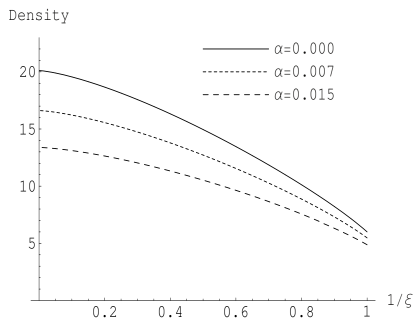

Figure 5. Density profiles for spherically symmetric ($m = 2$) flow; $\gamma = 7/5$.

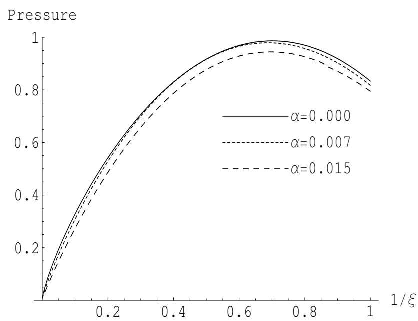

Figure 6. Pressure profiles for spherically symmetric ($m = 2$) flow; $\gamma = 7/5$.

We integrated the equations (5.6) numerically using the fourth order Runge–Kutta method for $1 \leq \xi < \infty$ and the values of velocity, density and pressure before collapse and at the instant of collapse are plotted in Figs. 1–6. The numerical results in the neighbourhood of $\xi = \infty$ are consistent with the results predicted by asymptotic relations (5.7).

Figures 1–6 show that behind the shock the velocity decreases and the density increases monotonically as we move towards the center/axis of collapse where $\xi \to \infty$; this increase in density behind the shock may be attributed to the geometrical convergence or the area contraction of the shock wave. The increase in density is further reinforced by a decrease in the value of α, and the decrease in velocity is further reinforced by an increase in the value of α. The behaviuor of the pressure is more complicated: the pressure profiles behind the shock exhibit non-monotonic variations. The pressure first increases, attains a maximum value and then decreases as we move towards the center/axis of collapse.

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