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# Quadratic/Linear Rational Spline Interpolation*

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**Abstract.** We describe the construction of an interpolating quadratic/linear rational spline $S$ of smoothness class $C^2$ for a strictly convex (or strictly concave) function $y$ on $[a,b]$. On uniform mesh $x_i = a + ih$, $i = 0, \dots, n$, in the case of sufficiently smooth function $y$ the expansions of $S$ and its derivatives are obtained. They give the superconvergence of order $h^4$ for the first derivative, of order $h^3$ for the second derivative and of order $h^2$ for the third derivative of $S$ in certain points. Corresponding numerical examples are given.

**Keywords:** rational spline, interpolation, superconvergence.

**AMS Subject Classification:** 65D07.

## 1 Introduction

For a strictly convex (or strictly concave) smooth function $y$ and interpolating quadratic/linear rational spline $S$ it is known that $\|S - y\|_\infty = O(h^4)$, see, e.g., [7, 8]. A quadratic/linear rational spline interpolant of class $C^2$ exists and is unique and strictly convex for any strictly convex data [10]. It should be effective to use these splines in seeking the solutions with singularities of differential and integral equations. As for nonconvex data such a rational spline interpolant cannot exist, an adaptive interpolation procedure is investigated in [11] which uses cubic polynomial and quadratic/linear rational pieces to retain strict convexity in the regions of strict convexity of data. The existence of such a coconvex spline interpolant is proved if data have weak alternation of second order divided differences on cubic sections. The problem of shape preserving interpolation has been considered by several authors [1, 2, 3, 4, 9, 12].

Quadratic/linear rational interpolating splines of class $C^2$ have the same accuracy as the classical cubic interpolating splines [8]. In some cases, the error is less for the cubic splines and in some cases, the error is less for the

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quadratic/linear rational splines. For the cubic splines, the expansions on subintervals via the derivatives of the smooth function to interpolate could be found, e.g., in [13]. For the linear/linear rational splines, such expansions could be found, e.g., in [5] and for quadratic splines, e.g., in [6]. They give the superconvergence of the spline values and its derivatives in certain points. We will study such a problem in the case of quadratic/linear rational spline interpolation. This needs expansions of a quadratic/linear rational spline interpolant with special boundary conditions and the establishment of them is the main purpose of our paper.

While the interpolation problem is a linear one, the quadratic/linear rational spline interpolation as well as linear/linear rational spline interpolation is, in nature, a nonlinear method because it leads to a nonlinear system with respect to the spline parameters. Nevertheless, the complexity of these rational spline interpolation methods is the same as in polynomial spline case.

## 2 Representation of Quadratic/Linear Rational Splines and Interpolation Problem

Consider a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \dots, n$, $h = (b-a)/n$, $n \in \mathbb{N}$. Quadratic/linear rational spline on each particular subinterval $[x_{i-1}, x_i]$ is a function $S$ of the form

$$S(x) = a_i + b_i(x - x_{i-1}) + \frac{c_i}{1 + d_i(x - x_{i-1})}, \quad x \in [x_{i-1}, x_i], \tag{2.1}$$

where $1 + d_i(x - x_{i-1}) > 0$. This gives for $x \in [x_{i-1}, x_i]$

$$S'(x) = b_i - \frac{c_i d_i}{(1 + d_i(x - x_{i-1}))^2}$$

and

$$S''(x) = \frac{2c_i d_i^2}{(1 + d_i(x - x_{i-1}))^3},$$

which means that $S$ or $-S$ is convex.

Using the notation $S(x_i) = S_i$ and $S''(x_i) = M_i$, $i = 0, \dots, n$, we get from (2.1)

$$\begin{aligned} S_{i-1} &= a_i + c_i, \quad S_i = a_i + b_i h + \frac{c_i}{1 + d_i h}, \\ M_{i-1} &= 2c_i d_i^2, \quad M_i = \frac{2c_i d_i^2}{(1 + d_i h)^3}. \end{aligned} \tag{2.2}$$

Consider at first the case $M_i \neq 0$. Then also $M_{i-1} \neq 0$ and $d_i \neq 0$. From (2.2) it follows

$$\begin{aligned} c_i &= \frac{M_{i-1}}{2d_i^2}, \quad a_i = S_{i-1} - \frac{M_{i-1}}{2d_i^2}, \\ b_i &= \frac{1}{h}(S_i - S_{i-1}) + \frac{M_{i-1}}{2d_i(1 + d_i h)}. \end{aligned}$$

Now the representation (2.1) is following

$$S(x) = S_{i-1} - \frac{M_{i-1}}{2d_i^2} + \left(\frac{S_i - S_{i-1}}{h} + \frac{M_{i-1}}{2d_i(1+d_ih)}\right)(x - x_{i-1})$$
$$+ \frac{M_{i-1}}{2d_i^2(1+d_i(x-x_{i-1}))}, \quad x \in [x_{i-1}, x_i]. \tag{2.3}$$

This gives for $x \in [x_{i-1}, x_i]$

$$S'(x) = \frac{S_i - S_{i-1}}{h} + \frac{M_{i-1}}{2d_i(1+d_ih)} - \frac{M_{i-1}}{2d_i(1+d_i(x-x_{i-1}))^2}, \tag{2.4}$$

$$S''(x) = \frac{M_{i-1}}{(1+d_i(x-x_{i-1}))^3}, \tag{2.5}$$

$$S'''(x) = -\frac{3d_iM_{i-1}}{(1+d_i(x-x_{i-1}))^4}. \tag{2.6}$$

While the continuity of $S$ and $S''$ is guaranteed by the representation (2.3), the continuity of $S'$, i.e., $S'(x_i - 0) = S'(x_i + 0)$, $i = 1, \ldots, n-1$, with the help of (2.4), leads to the equations

$$\frac{S_i - S_{i-1}}{h} + \frac{M_{i-1}h}{2(1+d_ih)^2} = \frac{S_{i+1} - S_i}{h} - \frac{M_ih}{2(1+d_{i+1}h)}.$$

From last two equations of (2.2) we get

$$1 + d_ih = \left(\frac{M_{i-1}}{M_i}\right)^{1/3}$$

and, thus, we have

$$M_i^{2/3}\left(M_{i-1}^{1/3} + M_{i+1}^{1/3}\right) = \frac{2}{h^2}(S_{i-1} - 2S_i + S_{i+1}), \quad i = 1, \ldots, n-1. \tag{2.7}$$

These interior equations of the quadratic/linear rational spline of class $C^2$ hold naturally in the case $M_i = 0$ (then $M_{i-1} = 0$ and $M_{i+1} = 0$) because then the spline is a linear function and (2.7) expresses the fact that its second order divided difference is equal to zero.

In interpolation problem, for given data $y_i$, $i = 0, \ldots, n$, we look for a spline $S$ such that

$$S(x_i) = y_i, \quad i = 0, \ldots, n. \tag{2.8}$$

In addition, we set the boundary conditions

$$S'(a) = \alpha_1, \quad S'(b) = \alpha_2 \tag{2.9}$$

or

$$S''(a) = \alpha_1, \quad S''(b) = \alpha_2 \tag{2.10}$$

for given $\alpha_1$ and $\alpha_2$, which we will specify later.

Actually, interpolating quadratic/linear rational spline is completely determined via the parameters $M_0, \ldots, M_n$. They could be found from a nonlinear system consisting of internal equations (2.7) where the values $S_0, \ldots, S_n$ are replaced from (2.8) and two boundary conditions from (2.9), (2.10) in different endpoints.

## 3 Second Moments of the Interpolant

In this section we study the nonlinear system with respect to the unknowns $M_0, \dots, M_n$.

Suppose that we have a sufficiently smooth function $y : [a, b] \to \mathbb{R}$ to interpolate. Denote $y_i = y(x_i)$, $i = 0, \dots, n$, similar notation will be used in the case of derivatives.

Let us write equations (2.7) with replaced values $S_i$ from (2.8) in the form

$$\begin{aligned} &\varphi_i(M_{i-1}, M_i, M_{i+1}) = M_i^{2/3}\big(M_{i-1}^{1/3} + M_{i+1}^{1/3}\big) - \frac{2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 0, \\ &\quad i = 1, \dots, n-1, \end{aligned} \tag{3.1}$$

introducing at the same time functions $\varphi_i$. Using at (3.1) the Taylor expansion and considering the boundary conditions (2.10) we have the system

$$\begin{cases} M_0 - \alpha_1 = 0, \\ \varphi_i(y''_{i-1}, y''_i, y''_{i+1}) + \dfrac{\partial \varphi_i}{\partial M_{i-1}}(y''_{i-1}, y''_i, y''_{i+1})(M_{i-1} - y''_{i-1}) \\ \quad + \dfrac{\partial \varphi_i}{\partial M_i}(y''_{i-1}, y''_i, y''_{i+1})(M_i - y''_i) \\ \quad + \dfrac{\partial \varphi_i}{\partial M_{i+1}}(y''_{i-1}, y''_i, y''_{i+1})(M_{i+1} - y''_{i+1}) + \dfrac{\varphi''_i}{2!}(\xi_\lambda)\bar{h}^2 = 0, \\ \qquad i = 1, \dots, n-1, \\ M_n - \alpha_2 = 0 \end{cases} \tag{3.2}$$

with the difference vector $\bar{h} = (M_{i-1} - y''_{i-1}, M_i - y''_i, M_{i+1} - y''_{i+1})$, some number $\lambda \in (0, 1)$ and $\xi_\lambda = (y''_{i-1}, y''_i, y''_{i+1}) + \lambda\bar{h}$. From (3.1) we calculate for $i = 1, \dots, n-1$

$$\begin{aligned} \frac{\partial \varphi_i}{\partial M_{i-1}}(M_{i-1}, M_i, M_{i+1}) &= \frac{1}{3}\left(\frac{M_i}{M_{i-1}}\right)^{2/3}, \\ \frac{\partial \varphi_i}{\partial M_i}(M_{i-1}, M_i, M_{i+1}) &= \frac{2}{3}\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3} + \left(\frac{M_{i+1}}{M_i}\right)^{1/3}\right), \\ \frac{\partial \varphi_i}{\partial M_{i+1}}(M_{i-1}, M_i, M_{i+1}) &= \frac{1}{3}\left(\frac{M_i}{M_{i+1}}\right)^{2/3}. \end{aligned} \tag{3.3}$$

Suppose in the following that $y \in C^4[a, b]$. We assume that $y''(x) > 0$ for all $x \in [a, b]$ or $y''(x) < 0$ for all $x \in [a, b]$ which means that $y$ or $-y$ is strictly convex. Let us expand $y_{i-1}$, $y_{i+1}$, $y''_{i-1}$ and $y''_{i+1}$ at the point $x_i$ by Taylor formula up to the forth derivative as

$$y_{i-1} = y_i - hy'_i + \frac{h^2}{2}y''_i - \frac{h^3}{6}y'''_i + \frac{h^4}{24}y_i^{IV} + o\big(h^4\big),$$

$$\begin{aligned}
y_{i+1} &= y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4),\\
y_{i-1}'' &= y_i'' - hy_i''' + \frac{h^2}{2}y_i^{IV} + o(h^2),\\
y_{i+1}'' &= y_i'' + hy_i''' + \frac{h^2}{2}y_i^{IV} + o(h^2).
\end{aligned}$$

First two expansions give us

$$\frac{2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 2y_i'' + \frac{1}{6}h^2 y_i^{IV} + o(h^2).$$

Then by (3.3) direct calculations yield

$$\begin{aligned}
\frac{\partial\varphi_i}{\partial M_{i-1}}(y_{i-1}'', y_i'', y_{i+1}'') &= \frac{1}{3} + \frac{2}{9}h\frac{y_i'''}{y_i''} - \frac{1}{9}h^2\frac{y_i^{IV}}{y_i''} + \frac{5}{27}h^2\left(\frac{y_i'''}{y_i''}\right)^2 + o(h^2),\\
\frac{\partial\varphi_i}{\partial M_i}(y_{i-1}'', y_i'', y_{i+1}'') &= \frac{4}{3} + \frac{2}{9}h^2\frac{y_i^{IV}}{y_i''} - \frac{4}{27}h^2\left(\frac{y_i'''}{y_i''}\right)^2 + o(h^2),\\
\frac{\partial\varphi_i}{\partial M_{i+1}}(y_{i-1}'', y_i'', y_{i+1}'') &= \frac{1}{3} - \frac{2}{9}h\frac{y_i'''}{y_i''} - \frac{1}{9}h^2\frac{y_i^{IV}}{y_i''} + \frac{5}{27}h^2\left(\frac{y_i'''}{y_i''}\right)^2 + o(h^2)
\end{aligned}$$

and also by (3.1)

$$\varphi_i(y_{i-1}'', y_i'', y_{i+1}'') = \frac{1}{6}h^2 y_i^{IV} - \frac{2}{9}h^2\frac{(y_i''')^2}{y_i''} + o(h^2)$$

which we replace in (3.2). We look for the solution of the obtained system such that

$$M_i = y_i'' + h^2\big[\psi(y)\big]_i + \beta_i, \quad i = 0, \dots, n,$$

where we suppose the function $\psi(y)$ to be continuous. Then

$$[\psi(y)]_{i-1} = [\psi(y)]_i + o(1), \quad [\psi(y)]_{i+1} = [\psi(y)]_i + o(1).$$

The entries in the matrix $\varphi_i''$ as second order partial derivatives of $\varphi_i$ could be calculated from (3.3). They contain a multiplier $M_j^{-1}$, $j = i-1, i, i+1$, of the expressions in (3.3) and are of order $O(1)$ provided we suppose, e.g., that $\beta_i = O(h)$. Then, in the case $\beta_i = O(h^2)$, due to the three-diagonality of the matrix $\varphi_i''$, we have $\varphi_i''(\xi_\lambda)\bar{h}^2 = O(h^4)$ and the system (3.2) could be written as

$$\begin{cases}
y_0'' + h^2\big[\psi(y)\big]_0 + \beta_0 - \alpha_1 = 0,\\
\frac{1}{6}h^2 y_i^{IV} - \frac{2}{9}h^2\frac{(y_i''')^2}{y_i''}\\
\qquad + \left(\frac{1}{3} + \frac{2}{9}h\frac{y_i'''}{y_i''} - \frac{1}{9}h^2\frac{y_i^{IV}}{y_i''} + \frac{5}{27}h^2\left(\frac{y_i'''}{y_i''}\right)^2\right)\left(h^2\big[\psi(y)\big]_i + \beta_{i-1}\right)
\end{cases}$$

$$
\begin{cases}
\quad + \left(\dfrac{4}{3} + \dfrac{2}{9}h^2\dfrac{y_i^{IV}}{y_i''} - \dfrac{4}{27}h^2\left(\dfrac{y_i'''}{y_i''}\right)^2\right)\left(h^2\left[\psi(y)\right]_i + \beta_i\right) \\
\quad + \left(\dfrac{1}{3} - \dfrac{2}{9}h\dfrac{y_i'''}{y_i''} - \dfrac{1}{9}h^2\dfrac{y_i^{IV}}{y_i''} + \dfrac{5}{27}h^2\left(\dfrac{y_i'''}{y_i''}\right)^2\right)\left(h^2\left[\psi(y)\right]_i + \beta_{i+1}\right) \\
\quad + o\bigl(h^2\bigr) = 0, \quad i = 1, \dots, n-1, \\
y_n'' + h^2\left[\psi(y)\right]_n + \beta_n - \alpha_2 = 0.
\end{cases} \tag{3.4}
$$

Determine the function $\psi(y)$ so that the coefficient at $h^2$ in interior equations is equal to 0. This gives

$$\psi(y) = -\frac{1}{12}\left(y^{IV} - \frac{4}{3}\frac{(y''')^2}{y''}\right).$$

Let us choose $\alpha_1$ and $\alpha_2$ so that $\beta_0 = o(h^2)$ and $\beta_n = o(h^2)$ (e.g., it may be $\beta_0 = \beta_n = 0$), thus, we pose the boundary conditions (2.10) in the form

$$
\begin{aligned}
S''(a) &= y''(a) - \frac{h^2}{12}\left(y^{IV}(a) - \frac{4}{3}\frac{(y'''(a))^2}{y''(a)}\right) + o\bigl(h^2\bigr), \\
S''(b) &= y''(b) - \frac{h^2}{12}\left(y^{IV}(b) - \frac{4}{3}\frac{(y'''(b))^2}{y''(b)}\right) + o\bigl(h^2\bigr).
\end{aligned} \tag{3.5}
$$

Finally, we get from (3.4) a system of the form $A\beta = \Phi(\beta)$ with respect to the unknowns $\beta = (\beta_0, \dots, \beta_n)$ having the matrix $A$ with diagonal dominance in rows and the components of $\Phi$ depending continuously on $\beta$. The equivalent system $\beta = A^{-1}\Phi(\beta)$ has a solution by Bohl–Brouwer fixed point principle because $A^{-1}\Phi$ maps a set $K = [-ch^2, ch^2]^{n+1}$ for some $c > 0$ into itself due to the fact that, for $\beta = O(h^2)$, we have $\Phi(\beta) = o(h^2)$. Recall that the solution of the interpolation problem is unique and, consequently, $\beta$ is uniquely determined. Thus, it holds $\beta_i = o(h^2)$, $i = 0, \dots, n$, and we arrive at the estimate

$$M_i = y_i'' - \frac{h^2}{12}\left(y_i^{IV} - \frac{4}{3}\frac{(y_i''')^2}{y_i''}\right) + o(h^2), \quad i = 0, \dots, n. \tag{3.6}$$

Note that in the case $y^{IV} \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, we have the error terms $O(h^{2+\alpha})$ instead of $o(h^2)$ in all earlier expansions and estimates.

## 4 Expansions of the Interpolant

In this section the expansions of interpolants on the whole particular interval will be established.

We still assume that $y \in C^4[a,b]$. In the interval $[x_{i-1}, x_i]$ let $x = x_{i-1} + th$, $t \in [0,1]$. Replacing $S_{i-1}$ and $S_i$ in (2.3) and (2.4) by $y_{i-1}$ and $y_i$, respectively, we write them in the form

$$S(x) = y_{i-1} - \frac{t(1-t)h^2 M_{i-1}}{2(1 + d_i h)(1 + d_i th)} \tag{4.1}$$

and

$$S'(x) = \frac{y_i - y_{i-1}}{h} + \frac{(t - 1 + t(1 + d_i th))hM_{i-1}}{2(1 + d_i h)(1 + d_i th)^2}. \tag{4.2}$$

Using also $1 + d_i h = (M_{i-1}/M_i)^{1/3}$ and (3.6) we establish with the help of Taylor formula the expansion

$$1 + d_i th = 1 + t\biggl(-\frac{h}{3}\frac{y_i'''}{y_i''} + h^2\biggl(\frac{1}{6}\frac{y_i^{IV}}{y_i''} - \frac{1}{9}\biggl(\frac{y_i'''}{y_i''}\biggr)^2\biggr)\biggr) + o\bigl(h^2\bigr).$$

This allows to express similarly $(1+d_i th)^2$, $(1+d_i th)^3$, $(1+d_i th)^4$ and $d_i$ needed in (4.1), (4.2), (2.5), (2.6). Finally, the Taylor expansion in $x \in [x_{i-1}, x_i]$ gives

$$S(x) = y(x) - \frac{t^2(1-t)^2}{24}h^4\biggl(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\biggr) + o\bigl(h^4\bigr), \tag{4.3}$$

$$S'(x) = y'(x) - \frac{t(1-t)(1-2t)}{12}h^3\biggl(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\biggr) + o\bigl(h^3\bigr), \tag{4.4}$$

$$S''(x) = y''(x) - \frac{1-6t(1-t)}{12}h^2\biggl(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\biggr) + o\bigl(h^2\bigr), \tag{4.5}$$

$$S'''(x) = y'''(x) + \frac{1-2t}{2}h\biggl(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\biggr) + o(h). \tag{4.6}$$

Note that (4.5) at $x = x_i$ coincides with (3.6).

We specified boundary conditions (2.10) by (3.5). Conditions (2.9) have to be used in the form

$$S'(a) = y'(a) + o\bigl(h^3\bigr), \qquad S'(b) = y'(b) + o\bigl(h^3\bigr). \tag{4.7}$$

Suppose that $y \in C^5[a,b]$. The reasoning of Section 3 gives then (3.6) with the rest term $o(h^3)$ instead of $o(h^2)$. Now we obtain

$$\begin{aligned}1 + d_i th = 1 + t\biggl(&-\frac{h}{3}\frac{y_i'''}{y_i''} + h^2\biggl(\frac{1}{6}\frac{y_i^{IV}}{y_i''} - \frac{1}{9}\biggl(\frac{y_i'''}{y_i''}\biggr)^2\biggr)\\ &+ h^3\biggl(-\frac{1}{36}\frac{y_i^{V}}{y_i''} + \frac{1}{108}\frac{y_i'''y_i^{IV}}{(y_i'')^2} + \frac{1}{81}\biggl(\frac{y_i'''}{y_i''}\biggr)^3\biggr)\biggr) + o\bigl(h^3\bigr)\end{aligned}$$

and then for $x \in [x_{i-1}, x_i]$

$$\begin{aligned}S(x) = y(x) &- \frac{t^2(1-t)^2}{24}h^4\biggl(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\biggr)\\ &- \frac{t(1-t)(1-2t)(1+3t(1-t))}{180}h^5\biggl(y^{V}(x) - \frac{10}{3}\frac{y'''(x)y^{IV}(x)}{y''(x)}\end{aligned}$$

$$+ \frac{20}{9}\frac{(y'''(x))^3}{(y''(x))^2}\Big) + o\big(h^5\big), \tag{4.8}$$

$$\begin{aligned} S'(x) = {}& y'(x) - \frac{t(1-t)(1-2t)}{12}h^3\left(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\right) \\ & - \frac{2-45t^2(1-t)^2}{360}h^4 y^V(x) + \frac{1-24t^2(1-t)^2}{54}h^4\frac{y'''(x)y^{IV}(x)}{y''(x)} \\ & - \frac{2-51t^2(1-t)^2}{162}h^4\frac{(y'''(x))^3}{(y''(x))^2} + o\big(h^4\big), \end{aligned} \tag{4.9}$$

$$\begin{aligned} S''(x) = {}& y''(x) - \frac{1-6t(1-t)}{12}h^2\left(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\right) \\ & + \frac{t(1-t)(1-2t)}{6}h^3\left(y^V(x) - 4\frac{y'''(x)y^{IV}(x)}{y''(x)} + \frac{28}{9}\frac{(y'''(x))^3}{(y''(x))^2}\right) \\ & + o\big(h^3\big), \end{aligned} \tag{4.10}$$

$$\begin{aligned} S'''(x) = {}& y'''(x) + \frac{1-2t}{2}h\left(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\right) \\ & + \frac{1-6t(1-t)}{12}h^2\left(y^V(x) - \frac{16}{3}\frac{y'''(x)y^{IV}(x)}{y''(x)} + \frac{44}{9}\frac{(y'''(x))^3}{(y''(x))^2}\right) \\ & + o\big(h^2\big). \end{aligned} \tag{4.11}$$

The boundary conditions (2.9) have to be specified now as

$$\begin{aligned} S'(a) &= y'(a) - h^4\left(\frac{1}{180}y^V(a) - \frac{1}{54}\frac{y'''(a)y^{IV}(a)}{y''(a)} + \frac{1}{81}\frac{(y'''(a))^3}{(y''(a))^2}\right) + o\big(h^4\big), \\ S'(b) &= y'(b) - h^4\left(\frac{1}{180}y^V(b) - \frac{1}{54}\frac{y'''(b)y^{IV}(b)}{y''(b)} + \frac{1}{81}\frac{(y'''(b))^3}{(y''(b))^2}\right) + o\big(h^4\big). \end{aligned} \tag{4.12}$$

We have proved the following

**Theorem 1.** *Let $y$ (or $-y$) be a strictly convex function. If $y \in C^4[a,b]$ then the quadratic/linear rational spline $S$ of smoothness class $C^2$ satisfying interpolation conditions* (2.8) *and boundary conditions* (3.5) *or* (4.7) *expands as shown in* (4.3)*–*(4.6)*. In the case $y \in C^5[a,b]$ the expansions* (4.8)*–*(4.11) *hold provided the boundary conditions* (3.5) *with the rest terms $o(h^3)$ instead of $o(h^2)$ or* (4.12) *are used.*

*Remark.* If $y^{IV} \in \operatorname{Lip}\alpha$ or $y^V \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, then in previous formulae all the rest terms written as $o(h^k)$ for some $k$ could be replaced by $O(h^{k+\alpha})$.

Basing on expansions (4.4)–(4.6) it is now immediate to obtain superconvergence assertions. From (4.4) we get $S'(x) = y'(x) + O(h^4)$ in points $x = x_i$ and $x = (x_{i-1}+x_i)/2$, (4.5) yields $S''(x) = y''(x) + O(h^3)$ in points $x = x_i + th$, corresponding to $t = (3 \pm \sqrt{3})/6$ and (4.6) gives $S'''(x) = y'''(x) + O(h^2)$ in points $(x_{i-1}+x_i)/2$.

Expansions for cubic spline interpolants were known earlier. They are given, e.g., in [13] in the case $y \in C^5[a,b]$, for $x \in [x_{i-1}, x_i]$

$$S(x) = y(x) - \frac{t^2(1-t)^2}{24} h^4 y^{IV}(x) - \frac{t(1-t)(1-2t)(1+3t(1-t))}{180} h^5 y^V(x) + o\big(h^5\big),$$

$$S'(x) = y'(x) - \frac{t(1-t)(1-2t)}{12} h^3 y^{IV}(x) - \frac{2-45t^2(1-t)^2}{360} h^4 y^V(x) + o\big(h^4\big),$$

$$S''(x) = y''(x) - \frac{1-6t(1-t)}{12} h^2 y^{IV}(x) + \frac{t(1-t)(1-2t)}{6} h^3 y^V(x) + o\big(h^3\big),$$

$$S'''(x) = y'''(x) + \frac{1-2t}{2} h y^{IV}(x) + \frac{1-6t(1-t)}{12} h^2 y^V(x) + o\big(h^2\big).$$

We see that the superconvergence takes place in the same points as well for quadratic/linear rational and cubic spline interpolants.

## 5 Numerical Examples

We interpolated the function $y(x) = x^{-2}$ on the interval $[-2, -0.2]$ by quadratic/ linear rational spline $S$ as described in Section 2. The boundary conditions (2.10) with

$$\alpha_1 = y_0 + \frac{2}{3} h^2 \frac{1}{x_0^6}, \qquad \alpha_2 = y_n + \frac{2}{3} h^2 \frac{1}{x_n^6}$$

were used. The "three-diagonal" nonlinear system (3.2) to determine the values $M_i$ was solved by Newton's method and the iterations were stopped at $\|M^k - M^{k-1}\|_\infty \le 10^{-10}$, $M^k$ being the sequence of approximations to the vector $M = (M_0, \dots, M_n)$. The errors $\varepsilon'_n = S'(z_i) - y'(z_i)$ and $\varepsilon'''_n = S'''(z_i) - y'''(z_i)$ were calculated in certain superconvergence points $z_i$. Results of numerical tests are presented in Tables 1–2.

**Table 1.** Numerical results for $\varepsilon'_n = S'(-1.1) - y'(-1.1)$.

| $n$ | 16 | 32 | 64 | 128 | 256 |
|---|---|---|---|---|---|
| $\varepsilon'_n$ | $1.1788 \cdot 10^{-5}$ | $7.5539 \cdot 10^{-7}$ | $4.7479 \cdot 10^{-8}$ | $2.9716 \cdot 10^{-9}$ | $1.8580 \cdot 10^{-10}$ |
| $\varepsilon'_{\frac{n}{2}} / \varepsilon'_n$ | | 15.6055 | 15.9101 | 15.9774 | 15.9938 |

We see from Tables 1 and 2 the superconvergence results predicted by theoretical estimates.

**Table 2.** Numerical results for $\varepsilon_n''' = S'''(z_i) - y'''(z_i)$, $i = 1, 2$.

| | $z_1 = \frac{a+b}{2} - \frac{h}{2}$ | | $z_2 = \frac{a+b}{2} + \frac{h}{2}$ | |
|---|---|---|---|---|
| $n$ | $\varepsilon_n'''$ | $\varepsilon_{\frac{n}{2}}'''/\varepsilon_n'''$ | $\varepsilon_n'''$ | $\varepsilon_{\frac{n}{2}}'''/\varepsilon_n'''$ |
| 16 | $-6.9813 \cdot 10^{-3}$ | | $-1.4140 \cdot 10^{-2}$ | |
| 32 | $-2.1037 \cdot 10^{-2}$ | 3.3186 | $-3.0075 \cdot 10^{-3}$ | 4.7017 |
| 64 | $-5.7679 \cdot 10^{-4}$ | 3.6473 | $-6.8979 \cdot 10^{-4}$ | 4.3600 |
| 128 | $-1.5091 \cdot 10^{-4}$ | 3.8221 | $-1.6504 \cdot 10^{-4}$ | 4.1796 |
| 256 | $-3.8583 \cdot 10^{-5}$ | 3.9113 | $-4.0362 \cdot 10^{-5}$ | 4.0889 |

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