Mathematical Modelling and Analysis
Volume 19 Number 1, February 2014, 66–74
http://dx.doi.org/10.3846/13926292.2014.893262

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

Limit Theorems for Twists of L-Functions of Elliptic Curves. IV

Virginija Garbaliauskienė[a] and Antanas Laurinčikas[b]

[a] *Šiauliai University*
P. Višinskio str. 19, LT-77156 Šiauliai, Lithuania
[b] *Vilnius University*
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
E-mail(*corresp.*): `virginija@fm.su.lt`
E-mail: `antanas.laurincikas@mif.vu.lt`

Received June 29, 2013; revised December 13, 2013; published online February 20, 2014

Abstract. In this paper, we prove a multidimensional limit theorem for moduli of twists of L-functions of elliptic curves. The limit measure in this theorem is defined by the characteristic transforms.

Keywords: Dirichlet character, elliptic curve, L-function of elliptic curve, twist of L-function of elliptic curve, weak convergence.

AMS Subject Classification: 11M41; 44K15.

1 Introduction

In [3], we have obtained a limit theorem for the modulus of twisted with Dirichlet character L-functions of elliptic curves with an increasing modulus of the character. Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$. For each prime number p, denote by E_p the reduction of the curve E modulo p which is a curve over the finite field $\mathbb{F}_p$, and define $\lambda(p)$ by the equality

$$\left|E(\mathbb{F}_p)\right| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ is the number of points of E_p. The L-function $L_E(s)$, $s = \sigma + it$, of the elliptic curve E is defined, for $\sigma > \frac{3}{2}$, by the product

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

and continues analytically to the whole complex plane.

Let χ be a Dirichlet character modulo q. The twist $L_E(s,\chi)$ with character χ for the function $L_E(s)$ is defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s,\chi) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}$$

and is analytically continued to an entire function.

Suppose that the modulus q of the character χ is a prime number, denote, as usual, by χ_0 the principal character modulo q, and, for $Q \geq 2$, define

$$M_Q = \sum_{q\leq Q}\sum_{\substack{\chi=\chi(\mathrm{mod}q)\\ \chi\neq\chi_0}} 1 \quad \text{and} \quad \mu_Q(\ldots) = M_Q^{-1}\sum_{q\leq Q}\sum_{\substack{\chi=\chi(\mathrm{mod}q)\\ \chi\neq\chi_0\\ \ldots}} 1,$$

where in place of dots we will write a condition satisfied by a pair $(q,\chi(\mathrm{mod}q))$. Moreover, denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. Then, in [3], for $\sigma > \frac{3}{2}$, we have proved a limit theorem on the weak convergence of the measure

$$\mu_Q\bigl(\left|L_E(s,\chi)\right| \in A\bigr), \quad A \in \mathcal{B}(\mathbb{R})$$

as $Q \to \infty$. The limit measure P in that theorem is defined by its characteristic transforms

$$w_k(\tau) = \int\limits_{\mathbb{R}\setminus\{0\}} |x|^{i\tau}\,\mathrm{sgn}^k\,\mathrm{d}P = \sum_{m=1}^{\infty}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}}, \quad \tau\in\mathbb{R},\ k=0,1,$$

where $a_\tau(m)$ and $b_\tau(m)$ are certain explicitly given multiplicative functions.

We note that the first theorems of the above type were obtained by P. Elliott [1] and [2], and E. Stankus [5] for Dirichlet L-functions.

The aim of this note is to prove a joint limit theorem for the moduli of the twists of L-functions of elliptic curves. For $j = 1,\ldots,r$, let E_j be an elliptic curve given by the Weierstrass equation

$$y^2 = x^3 + a_jx + b_j, \quad a_j, b_j \in \mathbb{Z}$$

with non-zero discriminant $\Delta_j = -16(4a_j^3 + 27b_j^2)$. Similarly as above, we define the quantities $\lambda_j(p)$, and the twist

$$L_{E_j}(s,\chi) = \prod_{p|\Delta_j}\left(1 - \frac{\lambda_j(p)\chi(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta_j}\left(1 - \frac{\lambda_j(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}, \quad \sigma > \frac{3}{2}.$$

This, for $\sigma > \frac{3}{2}$, can be rewritten in the form

$$L_{E_j}(s,\chi) = \prod_{p|\Delta_j}\left(1 - \frac{\lambda_j(p)\chi(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta_j}\left(1 - \frac{\alpha_j(p)\chi(p)}{p^s}\right)^{-1} \times\left(1 - \frac{\beta_j(p)\chi(p)}{p^s}\right)^{-1},$$

where $\alpha_j(p)$ and $\beta_j(p)$ are conjugate complex numbers such that $\alpha_j(p)+\beta_j(p) = \lambda_j(p)$ and $\alpha_j(p)\beta_j(p) = p$. For brevity, let $\eta = \eta(\tau) = \frac{i\tau}{2}$, $\tau \in \mathbb{R}$, and, for primes p and $k \in \mathbb{N}$, we set

$$d_\tau\bigl(p^k\bigr) = \frac{\eta(\eta+1)\cdots(\eta+k-1)}{k!}.$$

For $j = 1,\dots,r$, $p \nmid \Delta_j$ and $k \in \mathbb{N}$, we define

$$a_{j;\tau}\bigl(p^k\bigr) = \sum_{l=0}^{k} d_\tau\bigl(p^l\bigr)\alpha_j^l(p) d_\tau\bigl(p^{k-l}\bigr)\beta_j^{k-l}(p),$$
$$b_{j;\tau}\bigl(p^k\bigr) = \sum_{l=0}^{k} d_\tau\bigl(p^l\bigr)\overline{\alpha}_j^l(p) d_\tau\bigl(p^{k-l}\bigr)\overline{\beta}_j^{k-l}(p),$$

where $\overline{\alpha}_j(p)$ and $\overline{\beta}_j(p)$ denote the conjugates of $\alpha_j(p)$ and $\beta_j(p)$, respectively. For $j = 1,\dots,r$, $p \mid \Delta_j$ and $k \in \mathbb{N}$, we define

$$a_{j;\tau}\bigl(p^k\bigr) = b_{j;\tau}\bigl(p^k\bigr) = d_\tau\bigl(p^k\bigr)\lambda_j^k(p).$$

Now, for $j = 1,\dots,r$ and $m \in \mathbb{N}$, define

$$a_{j;\tau}(m) = \prod_{p^l \| m} a_{j;\tau}\bigl(p^l\bigr), \qquad b_{j;\tau}(m) = \prod_{p^l \| m} b_{j;\tau}\bigl(p^l\bigr),$$

where $p^l \parallel m$ means that $p^l \mid m$ but $p^{l+1} \nmid m$. Thus, $a_{j;\tau}(m)$ and $b_{j;\tau}(m)$ are multiplicative functions.

Now, on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, we will define a certain probability measure $P^{(r)}$. For this, we will use the characteristic transforms. Let P be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$. Denote by P_j, $j = 1,\dots,r$, P_{j_1,j_2}, $j_2 > j_1 = 1,\dots,r-1$, $\dots, P_{1,\dots,j-1,j+1,\dots,r}$, $j = 1,\dots,r$, the one-dimensional, two-dimensional, $\dots$, $(r-1)$-dimensional marginal measures of P, i.e.,

$$P_j(A) = P(\underbrace{\mathbb{R}\times\cdots\mathbb{R}}_{j-1}\times A\times\mathbb{R}\times\cdots\times\mathbb{R}), \quad A \in \mathcal{B}(\mathbb{R}),\ j = 1,\dots,r,$$
$$P_{j_1,j_2}(A_1\times A_2) = P(\underbrace{\underbrace{\mathbb{R}\times\cdots\mathbb{R}}_{j_1-1}\times A_1\times\mathbb{R}\times\cdots\times\mathbb{R}}_{j_2-1}\times A_2\times\mathbb{R}\times\cdots\times\mathbb{R}),$$
$$A_1, A_2 \in \mathcal{B}(\mathbb{R}),\ j_2 > j_1 = 1,\dots,r-1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_{1,\dots,j-1,j+1,\dots,r}(A_1\times\cdots\times A_{j-1}\times A_{j+1}\times\cdots\times A_r)$$
$$= P(A_1\times\cdots\times A_{j-1}\times\mathbb{R}\times A_{j+1}\times\cdots\times A_r),$$
$$A_1,\dots,A_{j-1},A_{j+1},\dots,A_r \in \mathcal{B}(\mathbb{R}),\ j = 1,\dots,r.$$

Then the functions

$$w_k(\tau) = \int_{\mathbb{R}\setminus\{0\}} |x|^{i\tau} \operatorname{sgn}^k x \, \mathrm{d}P_j, \quad \tau \in \mathbb{R},\ k = 0,1,\ j = 1,\dots,r,$$

$$w_{k_1,k_2}(\tau_1,\tau_2) = \int\limits_{\mathbb{R}\setminus\{0\}} \int\limits_{\mathbb{R}\setminus\{0\}} \big(|x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1\big)\big(|x_2|^{i\tau_2} \operatorname{sgn}^{k_2} x_2\big) \, \mathrm{d}P_{j_1,j_2},$$

$$\tau_1,\tau_2 \in \mathbb{R},\ k_1,k_2 = 0,1,\ j_2 > j_1 = 1,\dots,r,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_r)$$

$$= \int\limits_{\mathbb{R}\setminus\{0\}} \dots \int\limits_{\mathbb{R}\setminus\{0\}} \big(|x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1\big) \cdots \big(|x_{j-1}|^{i\tau_{j-1}} \operatorname{sgn}^{k_{j-1}} x_{j-1}\big)$$

$$\big(|x_{j+1}|^{i\tau_{j+1}} \operatorname{sgn}^{k_{j+1}} x_{j+1}\big) \cdots \big(|x_r|^{i\tau_r} \operatorname{sgn}^{k_r} x_r\big) \, \mathrm{d}P_{1,\dots,j-1,j+1,\dots,r},$$

$$\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_r \in \mathbb{R},\ k_1,\dots,k_{j-1},k_{j+1},\dots,k_r = 0,1,\ j = 1,\dots,r,$$

$$w_{k_1,\dots,k_r}(\tau_1,\dots,\tau_r) = \int\limits_{\mathbb{R}\setminus\{0\}} \dots \int\limits_{\mathbb{R}\setminus\{0\}} \big(|x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1\big) \cdots \big(|x_r|^{i\tau_r} \operatorname{sgn}^{k_r} x_r\big) \, \mathrm{d}P,$$

$$\tau_1,\dots,\tau_r \in \mathbb{R},\ k_1,\dots,k_r = 0,1$$

are called the characteristic transforms of the measure P. The characteristic transforms of multidimensional distribution functions were introduced in [4]. It is easily seen that the results of [4] remain valid when distribution functions are replaced by probability measures. Thus, we have that the measure P on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ is uniquely determined by its characteristic transforms $\{w_k(\tau), w_{k_1,k_2}(\tau_1,\tau_2), \dots, w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_r), w_{k_1,\dots,k_r}(\tau_1,\dots,\tau_r)\}$.

Now let $P^{(r)}$ be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ given by the following characteristic transforms:

$$w_k(\tau) = \sum_{m=1}^{\infty} \frac{a_{j;\tau}(m) b_{j;\tau}(m)}{m^{2\sigma_j}}, \quad j = 1,\dots,r,$$

$$w_{k_1,k_2}(\tau_1,\tau_2) = \sum_{m=1}^{\infty} \sum_{m_1 m_2 = m} \frac{a_{j_1;\tau_1}(m_1) a_{j_2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n_1 n_2 = m} \frac{b_{j_1;\tau_1}(n_1) b_{j_2;\tau_2}(n_2)}{n_1^{\overline{s}_1} n_2^{\overline{s}_2}},$$

$$j_2 > j_1 = 1,\dots,r-1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_r)$$

$$= \sum_{m=1}^{\infty} \sum_{m_1 \dots m_{j-1} m_{j+1} \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \cdots a_{j-1;\tau_{j-1}}(m_{j-1}) a_{j+1;\tau_{j+1}}(m_{j+1}) \cdots a_{r;\tau_r}(m_r)}{m_1^{s_1} \cdots m_{j-1}^{s_{j-1}} m_{j+1}^{s_{j+1}} \cdots m_r^{s_r}}$$

$$\times \sum_{n_1 \cdots n_{j-1} n_{j+1} \cdots n_r = m} \frac{b_{1;\tau_1}(n_1) \cdots b_{j-1;\tau_{j-1}}(n_{j-1}) b_{j+1;\tau_{j+1}}(n_{j+1}) \cdots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1} \cdots n_{j-1}^{\overline{s}_{j-1}} n_{j+1}^{\overline{s}_{j+1}} \cdots n_r^{\overline{s}_r}},$$

$$j = 1,\dots,r,$$

$$w_{k_1,\dots,k_r}(\tau_1,\dots,\tau_r) = \sum_{m=1}^{\infty} \sum_{m_1\cdots m_r=m} \frac{a_{1;\tau_1}(m_1)\cdots a_{r;\tau_r}(m_r)}{m_1^{s_1}\cdots m_r^{s_r}} \times \sum_{n_1\cdots n_r=m} \frac{b_{1;\tau_1}(n_1)\cdots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1}\cdots n_r^{\overline{s}_r}}. \tag{1.1}$$

Here $s_j = \sigma_j + it_j$, and $\sigma_j > \frac{3}{2}$, $j = 1,\dots,r$.

For $A \in \mathcal{B}(\mathbb{R}^r)$, define

$$P_Q(A) = \mu_Q\big(\big(|L_{E_1}(s_1,\chi)|,\dots,|L_{E_r}(s_r,\chi)|\big) \in A\big).$$

Theorem 1. *Suppose that* $\min_{1\le j\le r}\sigma_j > \frac{3}{2}$. *Then* P_Q *converges weakly to the probability measure* $P^{(r)}$ *as* $Q \to \infty$.

For the proof of Theorem 1, we will apply the method of characteristic transforms.

2 Characteristic Transforms of the Measure P_Q

In this section, we derive the formulae for the characteristic transforms $\{w_{j;Q}(\tau),\ w_{j_1,j_2;Q}(\tau_1,\tau_2),\ \dots,\ w_{1,\dots,j-1,j+1,\dots,r;Q}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_r),$ $w_Q(\tau_1,\dots,\tau_r)\}$ of the measure P_Q. Since P_Q is defined by means of the moduli of the twists $L_{E_j}(s,\chi)$, $j = 1,\dots,r$, its characteristic transforms do not depend on $k, k_1, k_2;\ \dots;\ k_1,\dots,k_r$. The definitions of P_Q and of characteristic transforms imply that

$$w_{j;Q}(\tau) = \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\bmod q)\\ \chi\ne\chi_0}} |L_{E_j}(s,\chi)|^{i\tau}, \quad j = 1,\dots,r,$$

$$w_{j_1,j_2;Q}(\tau_1,\tau_2) = \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\bmod q)\\ \chi\ne\chi_0}} |L_{E_{j_1}}(s_1,\chi)|^{i\tau_1}|L_{E_{j_2}}(s_2,\chi)|^{i\tau_2},$$

$$j_2 > j_1 = 1,\dots,r-1,$$

....

$$w_{1,\dots,j-1,j+1,\dots,r;Q}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_r) = \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\bmod q)\\ \chi\ne\chi_0}} |L_{E_1}(s_1,\chi)|^{i\tau_1},\dots,|L_{E_{j-1}}(s_{j-1},\chi)|^{i\tau_{j-1}} \times |L_{E_{j+1}}(s_{j+1},\chi)|^{i\tau_{j+1}}\cdots|L_{E_r}(s_r,\chi)|^{i\tau_r}, \quad j = 1,\dots,r,$$

$$w_Q(\tau_1,\dots,\tau_r) = \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\bmod q)\\ \chi\ne\chi_0}} |L_{E_1}(s_1,\chi)|^{i\tau_1}\cdots|L_{E_r}(s_r,\chi)|^{i\tau_r}. \tag{2.1}$$

Let δ be a fixed small positive number, and let $R_j = \{s \in \mathbb{C}\colon \sigma_j \ge \frac{3}{2} + \delta\}$. Then, in [3], it was obtained that

$$w_{j;Q}(\tau) = \sum_{m=1}^{\infty} \frac{a_{j;\tau}(m)b_{j;\tau}(m)}{m^{2\sigma_j}} + \mathrm{o}(1) \tag{2.2}$$

uniformly in $|\tau| \le c$ and $s_j \in R_j$ with arbitrary $c > 0$, as $Q \to \infty$, $j = 1, \dots, r$. Therefore, it remains to consider the characteristic transforms $w_{j_1,j_2;Q}(\tau_1, \tau_2)$, $\dots, w_{1,\dots,j-1,j+1,\dots,r;Q}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r), w_Q(\tau_1, \dots, \tau_r)$.

In [3], it was obtained that, for $s_j \in R_j$,

$$\left|L_{E_j}(s_j, \chi)\right|^{i\tau} = \sum_{m=1}^{\infty} \frac{\hat{a}_{j;\tau}(m)}{m^{s_j}} \sum_{n=1}^{\infty} \frac{\hat{b}_{j;\tau}(m)}{n^{\overline{s}_j}}, \tag{2.3}$$

where $\hat{a}_{j;\tau}(m)$ and $\hat{b}_{j;\tau}(m)$ are multiplicative functions given, for $p \nmid \varDelta_j$ and $k \in \mathbb{N}$, by

$$\hat{a}_{j;\tau}\left(p^k\right) = \sum_{l=0}^{k} d_\tau\left(p^l\right)\alpha_j^l(p)\chi\left(p^l\right)d_\tau\left(p^{k-l}\right)\beta_j^{k-l}(p)\chi\left(p^{k-l}\right) \tag{2.4}$$

$$\hat{b}_{j;\tau}\left(p^k\right) = \sum_{l=0}^{k} d_\tau\left(p^l\right)\overline{\alpha}_j^l(p)\overline{\chi}\left(p^l\right)d_\tau\left(p^{k-l}\right)\overline{\beta}_j^{k-l}(p)\overline{\chi}\left(p^{k-l}\right), \tag{2.5}$$

and, for $p \mid \varDelta_j$, $k \in \mathbb{N}$, by

$$\hat{a}_{j;\tau}(p^k) = d_k(p^k)\lambda_j^k(p)\chi^k(p), \quad \hat{b}_{j;\tau}(p^k) = d_\tau(p^k)\lambda_j^k(p)\overline{\chi}^k(p), \tag{2.6}$$

$j = 1, \dots, r$. Moreover, using the Hasse estimate

$$\left|\lambda_j(p)\right| \le 2\sqrt{p},$$

it was observed in [3] that, for $|\tau_j| \le c$,

$$\left|\hat{a}_{j;\tau_j}(m)\right| \le m^{\frac{1}{2}}d^{c_1}(m), \quad \left|\hat{b}_{j;\tau_j}(m)\right| \le m^{\frac{1}{2}}d^{c_1}(m), \tag{2.7}$$

where $d(m)$ denotes the divisor function, and c_1 is a positive constant depending only on c.

For simplicity, we will consider only the case $w_{1,2;Q}(\tau_1, \tau_2) \stackrel{def}{=} w_Q(\tau_1, \tau_2)$ and $w_Q(\tau_1, \dots, \tau_r)$. Other characteristic transforms of the measure P_Q are evaluated similarly. From (2.1) and (2.3), we find that, for $s_j \in R_j$,

$$\begin{aligned} w_Q(\tau_1, \tau_2) = \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \sum_{m_1=1}^{\infty} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \\ \times \sum_{n_1=1}^{\infty} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}} \sum_{m_2=1}^{\infty} \frac{\hat{a}_{2;\tau}(m_2)}{m_2^{s_2}} \sum_{n_2=1}^{\infty} \frac{\hat{b}_{2;\tau}(n_2)}{n_2^{\overline{s}_2}}, \end{aligned}$$

....

$$\begin{aligned} w_Q(\tau_1, \dots, \tau_r) = \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \sum_{m_1=1}^{\infty} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \\ \sum_{n_1=1}^{\infty} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}} \cdots \sum_{m_r=1}^{\infty} \frac{\hat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r=1}^{\infty} \frac{\hat{b}_{r;\tau_r}(n_r)}{n_r^{\overline{s}_r}}, \end{aligned} \tag{2.8}$$

where $\hat{a}_{j;\tau_j}(m)$ and $\hat{b}_{j;\tau_j}(m)$ are multiplicative functions defined by (2.4)–(2.6), and satisfying estimates (2.7), $j = 1, \dots, r$.

3 Asymptotics of Characteristic Transforms of P_Q

In this section, we consider the characteristic transforms of the measure P_Q as $Q \to \infty$. Let, for brevity, $N = \log Q$. Then the well-known estimate $d(m) = \mathrm{O}_\varepsilon(m^\varepsilon)$ with arbitrary $\varepsilon > 0$ together with estimates (2.7) shows that, for $|\tau_j| \le c$ and $s_j \in R_j$,

$$\sum_{m>N} \frac{\hat{a}_{j;\tau_j}(m)}{m^{s_j}} \ll \sum_{m>N} \frac{m^{\frac{1}{2}} d^{c_1}(m_j)}{m^{\frac{3}{2}+\delta}} \ll_\varepsilon \sum_{m>N} \frac{1}{m^{1+\delta-\varepsilon}} \ll_\varepsilon N^{-\delta+\varepsilon},$$
$$\sum_{n>N} \frac{\hat{b}_{j;\tau_j}(n)}{n_j^{\overline{s}}} = \ll_\varepsilon N^{-\delta+\varepsilon},$$

$j = 1, \dots, r$. Since, in view of (2.7), for $|\tau_j| \le c$ and $s_j \in R_j$,

$$\sum_{m\le N} \frac{\hat{a}_{j;\tau_j}(m)}{m^{s_j}} = \mathcal{O}(1) \quad \text{and} \quad \sum_{n\le N} \frac{\hat{b}_{j;\tau_j}(n)}{n^{\overline{s}_j}} = \mathcal{O}(1),$$

$j = 1, \dots, r$, from this and (2.8) we find that, in the above regions of τ_j and s_j,

$$w_Q(\tau_1, \tau_2) = \frac{1}{M_Q} \sum_{q\le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \Bigg(\sum_{m_1=\le N} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}}$$
$$\times \sum_{n_1\le N} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}} \sum_{m_2\le N}^{\infty} \frac{\hat{a}_{2;\tau_2}(m_2)}{m_2^{s_2}} \sum_{n_2\le N}^{\infty} \frac{\hat{b}_{2;\tau_2}(n_2)}{n_2^{\overline{s}_2}} \Bigg) + \mathcal{O}_\varepsilon\big(N^{-\delta+\varepsilon}\big),$$

. . . .

$$w_Q(\tau_1, \dots, \tau_r) = \frac{1}{M_Q} \sum_{q\le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \Bigg(\sum_{m_1\le N} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}}$$
$$\times \sum_{n_1\le N} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}} \cdots \sum_{m_r\le N}^{\infty} \frac{\hat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r\le N}^{\infty} \frac{\hat{b}_{r;\tau_r}(n_r)}{n_r^{\overline{s}_r}} \Bigg) + \mathcal{O}_\varepsilon\big(N^{-\delta+\varepsilon}\big). \quad (3.1)$$

The multiplicativity of the functions $\hat{a}_{j;\tau_j}(m)$ and $\hat{b}_{j;\tau_j}(m)$, and the complete multiplicativity of the character χ together with (2.4)–(2.6) and the definition of the multiplicative functions $a_{j;\tau_j}(m)$ and $b_{j;\tau_j}(m)$ show that

$$\hat{a}_{j;\tau_j}(m) = a_{j;\tau_j}(m)\chi(m), \quad \hat{b}_{j;\tau_j}(m) = b_{j;\tau_j}(m)\overline{\chi}(m), \quad (3.2)$$

$j = 1, \dots, r$. Therefore, the main terms on the right-hand sides of (3.1) are of the form

$$\sum_{m_1\le N} \frac{a_{1;\tau_1}(m_1)}{m_1^{s_1}} \sum_{n_1\le N} \frac{b_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}} \sum_{m_2\le N} \frac{a_{2;\tau_2}(m_2)}{m_2^{s_2}} \sum_{n_2\le N} \frac{b_{2;\tau_2}(n_2)}{n_2^{\overline{s}_2}}$$
$$\times \frac{1}{M_Q} \sum_{q\le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \chi(m_1 m_2)\overline{\chi}(n_1 n_2)$$

$$= \sum_{m \le N^2} \sum_{m_1 m_2 = m} \frac{a_{1;\tau_1}(m_1) a_{2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n \le N^2} \sum_{n_1 n_2 = n} \frac{b_{1;\tau_1}(n_1) b_{2;\tau_2}(n_2)}{n_1^{\overline{s}_1} n_2^{\overline{s}_2}} \times \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi(\bmod q) \\ \chi \ne \chi_0}} \chi(m) \overline{\chi}(n),$$

....

$$\sum_{m_1 \le N} \frac{a_{1;\tau_1}(m_1)}{m_1^{s_1}} \sum_{n_1 \le N} \frac{b_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}} \cdots \sum_{m_r \le N} \frac{a_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r \le N} \frac{b_{r;\tau_r}(n_r)}{n_r^{\overline{s}_r}} \times \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi(\bmod q) \\ \chi \ne \chi_0}} \chi(m_1 \cdots m_r) \overline{\chi}(n_1 \cdots n_r)$$

$$= \sum_{m \le N^r} \sum_{m_1 \cdots m_r = m} \frac{a_{1;\tau_1}(m_1) \cdots a_{r;\tau_r}(m_r)}{m_1^{s_1} \cdots m_r^{s_r}} \sum_{n \le N^r} \sum_{n_1 \cdots n_r = n} \frac{b_{1;\tau_1}(n_1) \cdots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1} \cdots n_r^{\overline{s}_r}} \times \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi(\bmod q) \\ \chi \ne \chi_0}} \chi(m) \overline{\chi}(n). \tag{3.3}$$

Consider two cases $m = n$ and $m \ne n$. If $m = n$, then

$$\sum_{q \le Q} \sum_{\substack{\chi = \chi(\bmod q) \\ \chi \ne \chi_0}} \chi(m) \overline{\chi}(n) = \sum_{q \le Q} \sum_{\substack{\chi = \chi(\bmod q) \\ \chi \ne \chi_0}} \bigl|\chi(m)\bigr|^2 = M_Q - \sum_{\substack{q \mid m \\ q \le N^j}} (q-2)$$

$$= M_Q + \mathcal{O}\biggl(\sum_{q \le N^j} q\biggr) = M_Q + \mathcal{O}\bigl(N^{2j}\bigr), \tag{3.4}$$

$j = 2, \ldots, r$. Moreover, in view of the estimate $\sum_{d_1 \cdots d_j = m} 1 = \mathcal{O}_\varepsilon\bigl(m^\varepsilon\bigr)$, $j=2,\ldots,r$, taking into account (2.7), (3.2), we deduce that, for $|\tau_j| \le c$ and $s_j \in R_j$,

$$\sum_{m_1 \cdots m_j = m} \frac{a_{1;\tau_1}(m_1) \cdots a_{j;\tau_j}(m_j)}{m_1^{s_1} \cdots m_j^{s_j}} \sum_{n_1 \cdots n_j = n} \frac{b_{1;\tau_1}(n_1) \cdots b_{j;\tau_j}(n_j)}{n_1^{\overline{s}_1} \cdots n_j^{\overline{s}_j}}$$

$$= \mathcal{O}\biggl(\sum_{m_1 \cdots m_j = m} \frac{d^{c_1}(m_1) \cdots d^{c_1}(m_j)}{m^{1+\delta}} \sum_{n_1 \cdots n_j = m} \frac{d^{c_1}(n_1) \cdots d^{c_1}(n_j)}{n^{1+\delta}}\biggr) = \mathcal{O}_\varepsilon\Bigl(\frac{1}{m^{2+2\delta-\varepsilon}}\Bigr).$$

Since [1] $M_Q = \frac{Q^2}{2 \log Q} + \mathcal{O}\Bigl(\frac{Q^2}{\log^2 Q}\Bigr)$, the latter estimate and (3.4) show that the case $m = n$ contributes to (3.1)

$$\sum_{m=1}^{\infty} \sum_{m_1 m_2 = m} \frac{a_{1;\tau_1}(m_1) a_{2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n_1 n_2 = m} \frac{b_{1;\tau_1}(n_1) b_{2;\tau_2}(n_2)}{n_1^{\overline{s}_1} n_2^{\overline{s}_2}} + \mathrm{o}(1),$$

....

$$\sum_{m=1}^{\infty} \sum_{m_1 \cdots m_r = m} \frac{a_{1;\tau_1}(m_1) \cdots a_{r;\tau_r}(m_r)}{m_1^{s_1} \cdots m_r^{s_r}} \sum_{n_1 \cdots n_r = m} \frac{b_{1;\tau_1}(n_1) \cdots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1} \cdots n_r^{\overline{s}_r}} + \mathrm{o}(1), \tag{3.5}$$

uniformly in $|\tau_j| \le c$ and $s_j \in R_j$, $j = 1, \dots, r$, as $Q \to \infty$.

Now suppose that $m \ne n$. Then, for $m, n \le N^j$, $j = 2, \dots, r$, similarly, as in [3], we obtain that

$$\sum_{q \le Q} \sum_{\substack{\chi = \chi(\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \chi(m)\overline{\chi}(n) = \mathcal{O}\left(\frac{Q}{\log Q}\right).$$

This, (3.1), (3.3) and (3.5) show that, uniformly in $|\tau_j| \le c$ and $s_j \in R_j$, $j = 1, \dots, r$,

$$w_Q(\tau_1, \tau_2) = \sum_{m=1}^{\infty} \sum_{m_1 m_2 = m} \frac{a_{1;\tau_1}(m_1) a_{2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n_1 n_2 = m} \frac{b_{1;\tau_1}(n_1) b_{2;\tau_2}(n_2)}{n_1^{\overline{s}_1} n_2^{\overline{s}_2}} + \mathrm{o}(1),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$w_Q(\tau_1, \dots, \tau_r) = \sum_{m=1}^{\infty} \sum_{m_1 \cdots m_r = m} \frac{a_{1;\tau_1}(m_1) \cdots a_{r;\tau_r}(m_r)}{m_1^{s_1} \cdots m_r^{s_r}} \times \sum_{n_1 \cdots n_r = m} \frac{b_{1;\tau_1}(n_1) \cdots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1} \cdots n_r^{\overline{s}_r}} + \mathrm{o}(1) \tag{3.6}$$

as $Q \to \infty$.

4 Proof of Theorem 1

By (2.2) and (3.6), we have that the characteristic transforms of the measure P_Q converge uniformly in $|\tau_j| \le c$ and $s_j \in R_j$, $j = 1, \dots, r$, to the functions defined by formulae (1.1) as $Q \to \infty$. The uniform convergence ensure the continuity of the functions (1.1). Thus, the application of a continuity theorem for characteristic transforms of the measures on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, Theorem 3 of [4], completes the proof of Theorem 1.

References

[1] P.D.T.A. Elliott. On the distribution of the values of L-series in the half-plane $\sigma > \frac{1}{2}$. *Indag. Math.*, **31**(3):222–234, 1971.

[2] P.D.T.A. Elliott. On the distribution of $\arg L(s, \chi)$ in the half-plane $\sigma > \frac{1}{2}$. *Acta Arith.*, **20**:155–169, 1972.

[3] V. Garbaliauskienė, A. Laurinčikas and E. Stankus. Limit theorems for twist of L-functions of elliptic curves. *Lith. Math. J.*, **50**(2):187–197, 2010.

[4] A. Laurinčikas. Multidimensional distribution of values of multiplicative functions. *Liet. Matem. Rink.*, **15**(2):13–24, 1975.

[5] E. Stankus. The distribution of L-function. *Liet. Matem. Rink.*, **15**(3):127–134, 1975.