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Using Chebyshev's polynomials for solving Fredholm integral equations of the second kind

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Abstract. The main problem with the Newton method is the computation of the inverse of the first derivative of the operator involved at each iteration step. Thus, when we want to apply the Newton method directly to solve an integral equation, the existence of the inverse of the first derivative is guaranteed, when the kernel is sufficiently differentiable into any of its two components, through its approximation by Taylor's polynomial. In this paper, we study the case in which the kernel is not differentiable in any of its two components. So, we present a strategy that consists of approximating the kernel of the nonlinear integral equation by a Chebyshev interpolation polynomial, which is separable. This allows us to explicitly calculate the inverse of the first derivative operator in each step of the Newton method and then approximate a solution of the approximate integral equation.

Keywords: Fredholm integral equation; the Newton method; existence domain; uniqueness domain.

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1 Introduction

Many problems that appear in Physics or Engineering are usually solved by methods for differential equations, but they can be solved, in some cases, more efficiently by methods of integral equations. For this, in recent years, papers have increasingly appeared for solving integral equations that provide methods for solving problems that until now could not be solved using standard methods of differential equations. These problems usually appear in applied mathematics, mathematical physics and other branches of science.

It is known that an integral equation is an equation in which the unknown function to be determined appears under the integral sign. In this work, we focus on the study of nonlinear integral equations of Fredholm of the second kind of the form

$$\phi(x) = f(x) + \lambda \int_a^b \mathcal{K}(x,t)\mathcal{H}(\phi)(t)\,dt, \quad x \in [a,b], \tag{1.1}$$

where λ is a given fixed real number, $f \in \mathcal{C}[a,b]$, the kernel $\mathcal{K}$ is a known function in $[a,b] \times [a,b]$, $\mathcal{H}$ is the Nemytskii operator $\mathcal{H} : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ such that $\mathcal{H}(\phi)(x) = H(\phi(x))$, where $H : \mathbb{R} \to \mathbb{R}$, and $\phi : [a,b] \to \mathbb{R}$ is the unknown function to be determined.

The numerical solution of nonlinear integral equations of the form (1.1) has two fundamental aspects. On the one hand, Equation (1.1) is discretized to obtain a solution $\phi(x)$ of the problem at the discretization points, so that we obtain a finite dimensional solution of the problem. Thus, we can find the projection methods, being the collocation and Garlekin methods the most common (see [10, 11]), as well as the Nystrom methods (see [17]).

On the other hand, there are iteration methods that, unlike the previous ones, approach the problem in an infinte-dimensional way, obtaining a function as a solution. Among these, the Picard method or fixed point iteration and Broyden's method stand out (see [3]), but the best known is the Newton method,

$$\phi_{n+1} = \phi_n - [\mathcal{T}'(\phi_n)]^{-1}\mathcal{T}(\phi_n), \quad n \geq 0, \quad \text{with } \phi_0 \text{ given}, \tag{1.2}$$

to solve the equation $\mathcal{T}(\phi) = 0$. The main problem with the Newton method is the calculation of $[\mathcal{T}'(\phi_n)]^{-1}$ in each step, which has been tried to solve by modifying the method in different ways [13, 14, 15]. In [7], the idea of constructing a Newton-type method by approximating $\mathcal{T}'(\phi_n)$ is used. For this, it is used the idea that the operator $[\mathcal{T}'(\phi_n)]^{-1}$ can be explicitly calculated when the integral kernel is separable, that is, there are real functions $\ell_i, \varpi_i \in \mathcal{C}[a,b]$ such that $\mathcal{K}(x,t) = \sum_{i=1}^{m} \ell_i(x)\varpi_i(t)$. Thus, when the kernel is sufficiently differentiable into one of its two components, we can approximate the kernel using a separable kernel by applying Taylor's polynomial on that component.

Following the previous idea, in this work, we approximate an equation $\mathcal{F}(\phi) = 0$, where $\mathcal{F} : \Omega \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ and

$$[\mathcal{F}(\phi)](x) = \phi(x) - f(x) - \lambda \int_a^b \mathcal{K}(x,t)\mathcal{H}(\phi)(t)\,dt, \quad x \in [a,b],$$

by an equation $\mathcal{G}(\phi) = 0$, so that, if ϕ^* is a solution of the equation $\mathcal{F}(\phi) = 0$ and $\widetilde{\phi}$ is a solution of the equation $\mathcal{G}(\phi) = 0$, the two solutions are as close as we want. For this, we consider $\mathcal{G} : \Omega \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ with

$$[\mathcal{G}(\phi)](x) = \phi(x) - f(x) - \lambda \int_a^b p_m(x,t)\mathcal{H}(\phi)(t)\,dt, \quad x \in [a,b], \tag{1.3}$$

where p_m is a polynomial of degree m and such that $\max\limits_{a \leq t \leq b} \left\{|\mathcal{K}(x,t) - p_m(x,t)|\right\}$, for each $x \in [a,b]$, is a small enough quantity. So, we can approximate a solution of the equation $\mathcal{F}(\phi) = 0$ in an infinite-dimensional way. To construct

p_m, we consider the interpolation polynomial for the function $\mathcal{K}(x,t)$, fixed $x \in [a,b]$, using Chebyshev nodes. In this way, we know that we avoid the well-known Runge phenomenon by increasing the number of nodes.

Besides, as Trefethen indicates in his paper [18], the convergence of the interpolation polynomial is guaranteed for interpolation in Chebyshev points so long as the function involved is somewhat smooth, e.g., Lipschitz continuous. More precisely, it is sufficient for the function to satisfy a Dini-Lipschitz condition [16]. Then, in this case, there exists $m \in \mathbb{N}$ such that $\max_{a\le t\le b}\left\{|\mathcal{K}(x,t)-p_m(x,t)|\right\}$ is as small as we want.

The paper is organized as follows. In Section 2, we present the problem statement, where we approximate the kernel $\mathcal{K}$ of the original integral equation (1.1) by a separable kernel using a polynomial p_m obtained from the interpolation of Chebyshev. In Section 3, we examine the convergence of the Newton method under a Lipschitz condition on the first derivative of the implicated operator and the technique of majorizing sequences. In Section 4, we obtain a result on the uniqueness of solution. In Section 5, we provide an algorithm of the Newton method that can be easily applied to the problem presented here. Finally, in Section 6, we illustrate the previous study with two examples.

Throughout the paper, we denote $\overline{B(\phi,\varrho)} = \{\nu \in \mathcal{C}[a,b]; \|\nu-\phi\| \le \varrho\}$, $B(\phi,\varrho) = \{\nu \in \mathcal{C}[a,b]; \|\nu-\phi\| < \varrho\}$ and the set of bounded linear operators from $\mathcal{C}[a,b]$ to $\mathcal{C}[a,b]$ by $\mathcal{L}(\mathcal{C}[a,b],\mathcal{C}[a,b])$, and use the infinity norm in $\mathcal{C}[a,b]$.

2 Problem statement

In [7], the authors impose the condition that the kernel is sufficiently differentiable in any of the components. In this work, we only require that the kernel be continuous in the two components. Thus, we can consider situations in which the kernel $\mathcal{K}$ is not sufficiently differentiable in either of its two components, that are not contemplated in [7]. For this, we use a Chebyshev interpolation polynomial, p_m, instead of a Taylor polynomial to approximate the kernel (as it is done in [7]).

Thus, it is known ([9, 12]) that given the i-th Chebyshev polynomial in $[-1,1]$,

$$T_i(t) = \cos\left(i \arccos t\right),$$

and the continuous kernel $\mathcal{K} : [a,b]\times[a,b] \to \mathbb{R}$, the Chebyshev interpolation polynomial that fits the data $\mathcal{K}(x,t_j)$, with $j = 0,1,\dots,m$, is given by

$$p_m(x,t) = \sum_{i=0}^{m} c_i(x) T_i\left(\frac{2t-b-a}{b-a}\right) = \sum_{i=0}^{m} c_i(x)\widetilde{T}_i(t), \tag{2.1}$$

where $\widetilde{T}_i(t) = T_i\left(\frac{2t-b-a}{b-a}\right)$ and

$$c_i(x) = \frac{2-\gamma_{i0}}{m+1}\sum_{j=0}^{m}\mathcal{K}(x,t_j)T_i\left(\frac{2t_j-b-a}{b-a}\right) = \frac{2-\gamma_{i0}}{m+1}\sum_{j=0}^{m}\mathcal{K}(x,t_j)\widetilde{T}_i(t_j),$$

with $\gamma_{00} = 1$, $\gamma_{i0} = 0$ if $i \neq 0$ and $t_j = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j+1}{2(m+1)}\,\pi\right)$ are the zeros of the polynomials of Chebyshev in $[a, b]$.

So, we consider the integral equation

$$\phi(x) = f(x) + \lambda \int_a^b p_m(x,t)\mathcal{H}(\phi)(t)\, dt, \quad x \in [a, b], \tag{2.2}$$

whose kernel is separable, and the operator $\mathcal{G}$ given in (1.3) with p_m defined in (2.1). Obviously, a zero of this $\mathcal{G}$ is a solution of (2.2). Besides, as the kernel p_m is separable, we can explicitly obtain $[\mathcal{G}'(\phi)]^{-1}$, so that we can apply the Newton method to approximate a solution of the equation $\mathcal{G}(\phi) = 0$ and, as a consequence, a solution of (2.2).

Evidently, our aim is to guarantee that both equations $\mathcal{F}(\phi) = 0$ and $\mathcal{G}(\phi) = 0$ are as close as we want. This depends on the adjustment we make of the kernel $\mathcal{K}$ through the polynomial p_m and this in turn depends on the Chebyshev polynomial considered. This fact has caused us to consider this type of interpolation because it smooths out the oscillations that occur in the interpolation polynomial by increasing its degree, also obtaining the convergence of Chebyshev's polynomial to the kernel $\mathcal{K}$ in the variable considered.

3 On the convergence of the Newton method

First, we need to be able to define $[\mathcal{G}'(\phi)]^{-1}$ in order to apply the Newton method to the operator (1.3) with p_m defined in (2.1), so that we need the kernel of (2.2) to be separable. As p_m is separable, we can write the operator $\mathcal{G}$ as

$$[\mathcal{G}(\phi)](x) = \phi(x) - f(x) - \lambda \sum_{i=0}^{m} c_i(x) \int_a^b \widetilde{T}_i(t)\mathcal{H}(\phi)(t)\, dt.$$

Moreover,

$$[\mathcal{G}'(\phi)\varphi](x) = \varphi(x) - \lambda \sum_{i=0}^{m} c_i(x) \int_a^b \widetilde{T}_i(t)[\mathcal{H}'(\phi)\varphi](t)\, dt. \tag{3.1}$$

So, we consider the Newton method,

$$\phi_{n+1} = \phi_n - [\mathcal{G}'(\phi_n)]^{-1}\mathcal{G}(\phi_n), \quad n \geq 0, \quad \text{with } \phi_0 \text{ given in } \mathcal{C}[a,b], \tag{3.2}$$

to approximate a solution of the equation $\mathcal{G}(\phi) = 0$. Then, our next aim is to prove the convergence of the Newton method.

For the convergence of the Newton method, we suppose that the first derivative of the Nemystkii operator is Lipschitz continuous in a domain $\Omega \subseteq \mathcal{C}[a, b]$,

$$\|\mathcal{H}'(\phi) - \mathcal{H}'(\varphi)\| \leq L\|\phi - \varphi\|, \quad L \geq 0, \quad \phi, \varphi \in \Omega. \tag{3.3}$$

Besides, given $\phi_0 \in \Omega$ and taking into account that $\left|\widetilde{T}_i(t)\right| \leq 1$, for $i = 0, 1, \ldots, m$, and

$$\|I - \mathcal{G}'(\phi_0)\| \leq |\lambda| C(b-a)\|\mathcal{H}'(\phi_0)\|,$$

where $C = \max_{a\le x\le b}\sum_{i=0}^{m}|c_i(x)|$, it follows, by the Banach lemma on invertible operators, that there exits $[\mathcal{G}'(\phi_0)]^{-1}$ such that $\|[\mathcal{G}'(\phi_0)]^{-1}\| \le \beta$ and $\|[\mathcal{G}'(\phi_0)]^{-1}\mathcal{G}(\phi_0)\| \le \eta$, where

$$\beta = \frac{1}{1 - Q\|\mathcal{H}'(\phi_0)\|}, \qquad \eta = \frac{\|\phi_0 - f\| + Q\|\mathcal{H}(\phi_0)\|}{1 - Q\|\mathcal{H}'(\phi_0)\|}, \tag{3.4}$$

with $Q = |\lambda|C(b-a)$ and provided that $\|\mathcal{H}'(\phi_0)\| < \frac{1}{Q}$.

After that, we see in the following result that $\mathcal{G}$ is a Lipschitz continuous operator in Ω. The proof is immediate from (3.3).

Lemma 1. *From* (3.1), (3.3) *and* $Q = |\lambda|C(b-a)$, *it follows*

$$\|\mathcal{G}'(\phi) - \mathcal{G}'(\varphi)\| \le QL\|\phi - \varphi\|, \text{ for } \phi,\varphi \in \Omega.$$

To prove the convergence of the Newton method (3.2), we use a modification of Kantorovich's method of majorizing sequences [4] and, in particular, the simple majorizing sequences defined in [5]. So, a sequence of positive real numbers $\{\alpha_n\}$ such that $\sum_{n\ge 0}\alpha_n = \alpha^* < +\infty$ is said to be a majorizing squence of the sequence $\{\phi_n\}$ if

$$\|\phi_{n+1} - \phi_n\| \le \alpha_n, \quad n \ge 0.$$

In addition, we remember the following result that is given in [5].

Theorem 1. *If* $\{\alpha_n\}$ *is a majorizing sequence of* (3.2), *then the sequence* (3.2) *converges to* $\widetilde{\phi}$ *and* $\phi_n, \widetilde{\phi} \in \overline{B(\phi_0,\alpha^*)}$ *with* $\alpha^* = \sum_{n\ge 0}\alpha_n$.

The next step is then to construct the majorizing sequence $\{\alpha_n\}$. For this, we consider the following auxiliary sequence of real numbers $\{\delta_n\}$:

$$\delta_0 = QL\beta\eta, \qquad \delta_n = \frac{\delta_{n-1}^2}{2(1-\delta_{n-1})^2}, \quad n \in \mathbb{N}, \tag{3.5}$$

and define the sequence of real numbers

$$\alpha_0 = \eta, \qquad \alpha_n = \frac{\delta_{n-1}\alpha_{n-1}}{2(1-\delta_{n-1})}, \quad n \in \mathbb{N}.$$

Besides, we have the next lemma that is used later.

Lemma 2. *If* $\phi_n, \phi_{n+1} \in \Omega$ *and* $\delta_n < 1$, *for all* $n \ge 0$, *we obtain, for* $n \in \mathbb{N}$, *the following recurrence relations:*

(I_n) $QL\|[\mathcal{G}'(\phi_{n-1})]^{-1}\|\|\phi_n - \phi_{n-1}\| \le \delta_{n-1}$,

(II_n) $\|[\mathcal{G}'(\phi_n)]^{-1}\| \le \dfrac{1}{1-\delta_{n-1}}\|[\mathcal{G}'(\phi_{n-1})]^{-1}\|$,

(III_n) $\|[\mathcal{G}(\phi_n)]\| \le \dfrac{1}{2}QL\|\phi_n - \phi_{n-1}\|^2$,

(IV_n) $\|\phi_{n+1} - \phi_n\| \leq \alpha_n$.

Proof. Item (I_1) is obvious. To prove (II_1), we observe that

$$\|I - [\mathcal{G}'(\phi_0)]^{-1}\mathcal{G}'(\phi_1)\| \leq QL\beta\|\phi_1 - \phi_0\| \leq \delta_0$$

and, provided that $\delta_0 < 1$, we have that the operator $[\mathcal{G}'(\phi_1)]^{-1}$ exists and $\|[\mathcal{G}'(\phi_1)]^{-1}\| \leq \frac{1}{1-\delta_0}\|[\mathcal{G}'(\phi_0)]^{-1}\|$ as a consequence of the Banach lemma on invertible operators.

From (3.2), we have

$$\mathcal{G}(\phi_0) + \mathcal{G}'(\phi_0)(\phi_1 - \phi_0) = 0$$

and, from the Taylor series,

$$\mathcal{G}(\phi_1) = \int_0^1 \left(\mathcal{G}'(\phi_0 + \tau(\phi_1 - \phi_0)) - \mathcal{G}'(\phi_0)\right)(\phi_1 - \phi_0)\, d\tau,$$

so that (III_1) follows from Lemma 1.

To prove (IV_1), we apply (II_1) and (III_1) as follows:

$$\|\phi_2 - \phi_1\| \leq \|[\mathcal{G}'(\phi_1)]^{-1}\|\|\mathcal{G}(\phi_1)\| \leq \frac{\delta_0}{2(1-\delta_0)}\|\phi_1 - \phi_0\| \leq \frac{\delta_0\alpha_0}{2(1-\delta_0)} = \alpha_1.$$

Finally, by mathematical induction on n, it follows (I_n)-(II_n)-(III_n)-(IV_n) for all $n \in \mathbb{N}$. □

We have just seen in the previous lemma that the restriction $\delta_n < 1$, for all $n \in \mathbb{N}$, is required on the sequence $\{\delta_n\}$ and we also have to guarantee that $\sum_{n\geq 0} \alpha_n = \alpha^* < +\infty$. In the following result we study the real sequences of positive numbers $\{\delta_n\}$ and $\{\alpha_n\}$.

Lemma 3. *If* $\delta_0 < \frac{1}{2}$, *then*

(a) *the sequence* $\{\delta_n\}$ *is decreasing,*

(b) $\alpha_n \leq \left(\frac{\delta_0}{2(1-\delta_0)}\right)^n \alpha_0$,

(c) $\sum_{n\geq 0} \alpha_n \leq \sum_{n\geq 0} \left(\frac{\delta_0}{2(1-\delta_0)}\right)^n \alpha_0 = \frac{2(1-\delta_0)\alpha_0}{2-3\delta_0}$.

Proof. First, we use mathematical induction. As $\delta_0 < \frac{1}{2}$, then $\delta_1 < \delta_0$. If $\delta_k < \delta_{k-1}$, for $k = 0, 1, \ldots, n-1$, then $\delta_n < \delta_{n-1}$, since $\delta_n = \frac{h(\delta_{n-1})^2}{2}$ with $h(t) = \frac{t}{1-t}$ and $h(t) > 0$ and $h'(t) > 0$ in $\left[0, \frac{1}{2}\right)$.

Second,

$$\alpha_n = \left(\frac{\delta_{n-1}}{2(1-\delta_{n-1})}\right)\alpha_{n-1} \leq \left(\frac{\delta_0}{2(1-\delta_0)}\right)\alpha_{n-1} \leq \cdots \leq \left(\frac{\delta_0}{2(1-\delta_0)}\right)^n \alpha_0.$$

Third,

$$\sum_{n\geq 0} \alpha_n \leq \sum_{n\geq 0} \left(\frac{\delta_0}{2(1-\delta_0)}\right)^n \alpha_0 = \frac{2(1-\delta_0)\alpha_0}{2-3\delta_0},$$

since $\frac{\delta_0}{2(1-\delta_0)} < 1$. □

Then, by the last lemma, we have a sequence of positive real numbers $\{\alpha_n\}$ with $\sum_{n\geq 0} \alpha_n = \alpha^* = \frac{2(1-\delta_0)\alpha_0}{2-3\delta_0}$ and, by Lemma 2, such that

$$\|\phi_n - \phi_{n-1}\| \leq \alpha_n \leq \left(\frac{\delta_0}{2(1-\delta_0)}\right)^n \alpha_0.$$

Therefore, by Theorem 1, the Newton sequence (3.2) converges to $\widetilde{\phi}$ if $\{\phi_n\} \subset \Omega$ and $\delta_0 < \frac{1}{2}$. So, we can prove the next result on the convergence of the Newton sequence $\{\phi_n\}$.

Theorem 2. *Let $\mathcal{G} : \Omega \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$, where Ω is an open convex non-empty set. If the condition* (3.3) *holds, there exists some $\phi_0 \in \Omega$ such that $\|\mathcal{H}'(\phi_0)\| \leq \frac{1}{Q}$ and $\delta_0 < \frac{1}{2}$, where β and η are given in* (3.4)*, and $B(\phi_0, \alpha^*) \subset \Omega$ with $\alpha^* = \frac{2(1-\delta_0)\alpha_0}{2-3\delta_0}$, then the Newton sequence, given by* (3.2)*, is convergent to a solution $\widetilde{\phi}$ of $\mathcal{G}(\phi) = 0$. Moreover, $\phi_n, \widetilde{\phi} \in \overline{B(\phi_0, \alpha^*)}$, for all $n \in \mathbb{N}$.*

Proof. On the one hand, we observe that

$$\|\phi_n - \phi_0\| \leq \sum_{i=0}^{n-1} \|\phi_{i+1} - \phi_i\| < \sum_{i=0}^{n-1} \alpha_i < \frac{2(1-\delta_0)}{2-3\delta_0}\,\eta = \alpha^*,$$

for all $n \in \mathbb{N}$. Therefore, $\{\phi_n\} \subset \Omega$. On the other hand, as $\delta_0 < \frac{1}{2}$, there exists $\widetilde{\phi}$ such that $\widetilde{\phi} = \lim_n \phi_n$. In addition, from the item (III_n) of Lemma 2, we have $\mathcal{G}(\widetilde{\phi}) = 0$ by continuity, since $\lim_n \|\phi_{n+1} - \phi_n\| = 0$. □

Next, we prove the quadratic convergence of the Newton method (3.2). For this, we use the concept of R-order of convergence [4].

Theorem 3. *Under the conditions of Theorem 2, the Newton sequence* (3.2) *hast R-order of convergence at least two.*

Proof. Consider the real functions $u(t) = \frac{1}{1-t}$ and $v(t) = \frac{t}{2}$. Then, from (3.5), we can write $\delta_0 = QL\beta\eta$, $\delta_n = \delta_{n-1}u(\delta_{n-1})^2 v(\delta_{n-1})$, $n \geq 1$. If we denote $b = \frac{\delta_1}{\delta_0}$ and take into account $\delta_0 < \frac{1}{2}$, it follows

$$\delta_n < b^{2^{n-1}}\delta_{n-1} \quad \text{and} \quad \delta_n < b^{2^n-1}\delta_0, \quad \text{for} \quad n \geq 2.$$

Observe also that $b < 1$, since $u(\delta_0)^2 v(\delta_0) < 1$ as a consequence of $\delta_0 < \frac{1}{2}$. Indeed, we prove the last by mathematical induction on n. As u is an increasing function, we have

$$\delta_2 = \delta_1 u(\delta_1)^2 v(\delta_1) = b\delta_0 u(b\delta_0)^2 v(b\delta_0) < b^2\delta_1 = b^3\delta_0.$$

Next, we suppose that $\delta_{n-1} < b^{2^{n-2}}\delta_{n-2} < b^{2^n-1}\delta_0$. Then,

$$\begin{aligned}\delta_n =&\delta_{n-1}u(\delta_{n-1})^2 v(\delta_{n-1}) < b^{2^{n-2}}\delta_{n-2}\ u\left(b^{2^{n-2}}\delta_{n-2}\right)^2 v\left(b^{2^{n-2}}\delta_{n-2}\right)\\ <&b^{2^{n-1}}\delta_{n-1} < b^{2^{n-1}}b^{2^{n-2}}\delta_{n-2} < \cdots < b^{2^n-1}\delta_0.\end{aligned}$$

After that, if we proceed as in Lemma 2, it is easy to prove by mathematical induction that the following holds:

$$\|\phi_n - \phi_{n-1}\| \leq u(\delta_{n-1})v(\delta_{n-1})\|\phi_{n-1} - \phi_{n-2}\|, \quad n \geq 2.$$

In addition, for $m \geq 1$ and $n \geq 1$, we obtain

$$\begin{aligned}&\|\phi_{n+m} - \phi_n\| \leq \sum_{i=n}^{n+m-1} \|\phi_{i+1} - \phi_i\|\\ &\leq \left(1 + \sum_{i=n}^{n+m-2}\left(\prod_{j=n}^{i} u(\delta_j)v(\delta_j)\right)\right)\|\phi_{n+1} - \phi_n\|\\ &\leq \sum_{i=n-1}^{n+m-2}\left(\prod_{j=0}^{i} u(\delta_j)v(\delta_j)\right)\|\phi_1 - \phi_0\|\\ &< \sum_{i=n-1}^{n+m-2}\left(\prod_{j=0}^{i} b^{2^j-1}u(\delta_0)v(\delta_0)\right)\|\phi_1 - \phi_0\| = \sum_{i=n-1}^{n+m-2}\left(\prod_{j=0}^{i}\left(b^{2^j}\Delta\right)\right)\|\phi_1 - \phi_0\|\\ &= \sum_{i=n-1}^{n+m-2}\left(b^{2^{1+i}-1}\Delta^{1+i}\right)\|\phi_1 - \phi_0\| = \sum_{i=0}^{m-1}\left(b^{2^{n+i}-1}\Delta^{n+i}\right)\|\phi_1 - \phi_0\|,\end{aligned}$$

where $\Delta = 1 - \delta_0 < 1$. Now, by applying the Bernouilli inequality, $b^{2^{n+i}-1} = b^{2^n-1}b^{2^n(2^i-1)} \leq b^{2^n-1}b^{2^n i}$, it follows

$$\|\phi_{n+m}-\phi_n\|<\left(\sum_{i=0}^{m-1} b^{2^n i}\Delta^i\right)b^{2^n-1}\Delta^n\|\phi_1-\phi_0\|<\frac{1-\left(b^{2^n}\Delta\right)^m}{1-b^{2^n}\Delta}\ b^{2^n-1}\Delta^n\|\phi_1-\phi_0\|.$$

So, as $m \to \infty$, we have

$$\|\phi^* - \phi_n\| < \left(b^{2^n-1}\right)\frac{\Delta^n}{1 - b^{2^n}\Delta}\|\phi_1 - \phi_0\| < b^{2^n}\frac{\|\phi_1 - \phi_0\|}{b(1 - \Delta)}$$

and then the R-quadratic convergence of the method (3.2) follows. □

4 Uniqueness of solution

Once a solution of the equation $\mathcal{G}(\phi) = 0$ is located by Theorem 2 in $\overline{B(\phi_0, \alpha^*)}$, we now separate it from other possible solutions by the following result of uniqueness of solution.

Theorem 4. *Under the hypotheses of Theorem 2, the solution* $\widetilde{\phi}$ *of* $\mathcal{G}(\phi) = 0$ *is unique in* $B(\phi_0, r) \cap \Omega$*, where* $r = \frac{2}{L}\left(\frac{1}{|\lambda|C(b-a)} - \|\mathcal{H}'(\phi_0)\|\right) - \alpha^*$ *and* $C = \max_{a\leq x\leq b}\sum_{i=0}^{m}|c_i(x)|$.

Proof. We suppose that ζ is another solution of the equation $\mathcal{G}(\phi) = 0$ in $B(\phi_0, r) \cap \Omega$. Thus,

$$0 = \mathcal{G}(\zeta) - \mathcal{G}(\widetilde{\phi}) = \left(\int_0^1 \mathcal{G}'(\widetilde{\phi} + \tau(\zeta - \widetilde{\phi}))\, d\tau \right) (\zeta - \widetilde{\phi}) = \Psi(\zeta - \widetilde{\phi}).$$

Now, if the operator $\Psi = \int_0^1 \mathcal{G}'(\widetilde{\phi} + \tau(\zeta - \widetilde{\phi}))\, d\tau$ is invertible, then $\zeta = \widetilde{\phi}$. We can deduce the last from the Banach lemma on invertible operators, provided that $\|I - \Psi\| < 1$. Observe that

$$\begin{aligned}
[(I - \Psi)z](x) &= \left(\left(\int_0^1 (I - \mathcal{G}'(\widetilde{\phi} + \tau(\zeta - \widetilde{\phi})))d\tau \right) z \right)(x) \\
&= \lambda \sum_{i=0}^{m} c_i(x) \left(\int_0^1 \left(\int_a^b \widetilde{T}_i(t)[\mathcal{H}'(\widetilde{\phi} + \tau(\zeta - \widetilde{\phi}))z](t)\, dt \right) d\tau \right), \\
\|I - \Psi\| &\le |\lambda| C \int_0^1 \int_a^b \|\mathcal{H}'(\widetilde{\phi} + \tau(\zeta - \widetilde{\phi}))\|\, dt\, d\tau \\
&< |\lambda| C (b-a) \int_0^1 \left(\|\mathcal{H}'(\phi_0)\| + L(\alpha^* + \tau(r - \alpha^*)) \right) d\tau \\
&= |\lambda| C (b-a) \left(\|\mathcal{H}'(\phi_0)\| + \frac{L}{2}(r + \alpha^*) \right) = 1
\end{aligned}$$

and the proof is then complete. □

Besides, we obtain that the solution is unique in $\overline{B(\phi_0, \alpha^*)}$ provided that $|\lambda| C(b-a) \left(\|\mathcal{H}'(\phi_0)\| + L\alpha^* \right) < 1$.

5 A simple algorithm of the Newton method

Our next aim is to find a simple algorithm to apply the Newton method to the operator $\mathcal{G}$. This is possible due to the separable approximation that we make of the integral kernel $\mathcal{K}$ by means of the Chebyshev interpolation polynomial p_m, since the latter allows us to define an operator $\mathcal{G}$ whose integral kernel is separable.

So, taking into account (3.1), if $[\mathcal{G}'(\phi)\varphi](x) = w(x)$, then,

$$\varphi(x) = \left(\mathcal{G}'(\phi)^{-1} w \right)(x) = w(x) + \lambda \sum_{i=0}^{m} c_i(x) \mathcal{I}_i, \tag{5.1}$$

where $\mathcal{I}_i = \int_a^b \widetilde{T}_i(t)[\mathcal{H}'(\phi)\varphi](t)dt$. Note that we can obtain $\mathcal{I}_i$, for $i = 0, 1, \ldots, m$, based on ϕ and w. For this, we map $\widetilde{T}_k(x)\mathcal{H}'(\phi)$ in (5.1) and integrate as follows:

$$\begin{aligned}
\int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi)\varphi](x)\, dx = &\int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi)w](x)\, dx \\
&+ \lambda \sum_{i=0}^{m} \int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi)c_i](x)\, dx\, \mathcal{I}_i.
\end{aligned}$$

Thus,

$$\mathcal{I}_k = b_k(\phi, w) + \lambda \sum_{i=0}^{m} a_{ki}(\phi)\, \mathcal{I}_i, \tag{5.2}$$

where

$$a_{ki}(\phi) = \int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi)c_i](x)\,dx, \quad b_k(\phi, w) = \int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi)w](x)\,dx.$$

Observe that (5.2), with $k = 0, 1, \ldots, m$, is the linear system of $m+1$ equations with $m+1$ unknowns given by

$$\begin{pmatrix} 1-\lambda a_{00}(\phi) & -\lambda a_{01}(\phi) & -\lambda a_{02}(\phi) & \cdots & -\lambda a_{0m}(\phi) \\ -\lambda a_{10}(\phi) & 1-\lambda a_{11}(\phi) & -\lambda a_{12}(\phi) & \cdots & -\lambda a_{1m}(\phi) \\ -\lambda a_{20}(\phi) & -\lambda a_{21}(\phi) & 1-\lambda a_{22}(\phi) & \cdots & -\lambda a_{2m}(\phi) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda a_{m0}(\phi) & -\lambda a_{m1}(\phi) & -\lambda a_{m2}(\phi) & \cdots & 1-\lambda a_{mm}(\phi) \end{pmatrix} \begin{pmatrix} \mathcal{I}_0 \\ \mathcal{I}_1 \\ \mathcal{I}_2 \\ \vdots \\ \mathcal{I}_m \end{pmatrix} = \begin{pmatrix} b_0(\phi, w) \\ b_1(\phi, w) \\ b_2(\phi, w) \\ \vdots \\ b_m(\phi, w) \end{pmatrix}, \tag{5.3}$$

that is represented in matrix form by

$$\mathbb{M}(\phi)\mathbb{I}^T = \mathbb{b}(\phi, w)^T,$$

where $\mathbb{I} = (\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_m)$ and $\mathbb{b}(\phi, w) = (b_0(\phi, w), b_1(\phi, w), \ldots, b_m(\phi, w))$. Therefore, if $\mathbb{M}(\phi)^{-1}$ exists, then there exists a unique solution of the lineal system (5.3) and we can write

$$\left(\mathcal{G}'(\phi)^{-1}w\right)(x) = w(x) + \lambda \mathbb{C}(x)\mathbb{I}^T,$$

with $\mathbb{C}(x) = (c_0(x), c_1(x), \ldots, c_m(x))$ and $x \in [a, b]$.

Taking into account the last, we can consider the following simple algorithm to apply the Newton method to approximate a solution of the equation $\mathcal{G}(\phi) = 0$:

Step 1. Given $\phi_0 \in \Omega$ such that $\|\mathcal{H}'(\phi_0)\| \leq \frac{1}{Q}$ and $\delta_0 = QL\beta\eta < \frac{1}{2}$, calculate

$$d_i(\phi_n) = \int_a^b \widetilde{T}_i(t)\mathcal{H}(\phi_n)(t)\,dt, \quad \text{for } i = 0, 1, \ldots, m,$$
$$\mathcal{G}(\phi_n)(x) = \phi_n(x) - f(x) - \lambda \mathbb{C}(x)\mathbb{D}(\phi_n)^T, \quad x \in [a, b],$$
with $\mathbb{D}(\phi_n) = (d_0(\phi_n), d_1(\phi_n), \ldots, d_m(\phi_n))$.

Step 2. Calculate

$$a_{ki}(\phi_n) = \int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi_n)c_i](x)\,dx, \quad 0 \leq k, i \leq m,$$

$$b_k(\phi_n, \mathcal{G}(\phi_n)) = \int_a^b \widetilde{T}_k(x)[\mathcal{H}'(\phi_n)\mathcal{G}(\phi_n)](x)\,dx, \quad 0 \leq k \leq m.$$

Step 3. Solve (5.3) for $\phi = \phi_n$.

Step 4. Calculate

$$\phi_{n+1}(x) = \phi_n(x) - \mathcal{G}(\phi_n)(x) - \lambda\mathbb{C}(x)\mathbb{I}^T, \quad x \in [a, b].$$

Notice that unlike the methods of discretization used by other authors, in this case, the iterations are continuous functions that approximate a solution of the equation $\mathcal{G}(\phi) = 0$.

Besides, as we have indicated previously, if the matrix $\mathbb{M}(\phi_n)^{-1}$ exists, the linear system (5.3) for $\phi = \phi_n$ has the solution $\mathbb{I}$. We see below that the existence of $\mathbb{M}(\phi_n)^{-1}$ is guaranteed under the condition $\|\mathcal{H}'(\phi_0)\| < \frac{1}{Q}$ required to the starting function ϕ_0. First, we introduce the following technical lemma.

Lemma 4. *If $\|\mathcal{H}'(\phi_0)\| < 1/Q$, for $\phi_0 \in \Omega$, then there exists $\mathbb{M}(\phi_0)^{-1}$ and $\|\mathbb{M}(\phi_0)^{-1}\| \leq \frac{1}{1-Q\|\mathcal{H}'(\phi_0)\|} = \beta$.*

Proof. As $\|\mathcal{H}'(\phi_0)\| < 1/Q$, it follows

$$\begin{aligned}
&\|I - \mathbb{M}(\phi_0)\| \leq |\lambda| \max_{0\leq k\leq m} \sum_{i=0}^{m} |a_{ki}(\phi_0)| \\
&\quad \leq |\lambda| \max_{0\leq k\leq m} \sum_{i=0}^{m} \int_a^b \left|\widetilde{T}_k(x)\right| \left|[\mathcal{H}'(\phi_0)c_i](x)\right| dx \leq |\lambda| C(b-a)\|\mathcal{H}'(\phi_0)\| < 1.
\end{aligned}$$

Therefore, $\mathbb{M}(\phi_0)^{-1}$ exists and $\|\mathbb{M}(\phi_0)^{-1}\| \leq \beta$. $\square$

Now, we are ready to prove the existence of the matrix $\mathbb{M}(\phi_n)^{-1}$, for all $n \geq 0$.

Theorem 5. *Under the hypotheses of Theorem 2, there exists $\mathbb{M}(\phi_n)^{-1}$, for all $n \geq 0$.*

Proof. From Lemma 4 and

$$\begin{aligned}
\|I - \mathbb{M}(\phi_0)^{-1}\mathbb{M}(\phi_1)\| &\leq \|\mathbb{M}(\phi_0)^{-1}\|\|\mathbb{M}(\phi_0) - \mathbb{M}(\phi_1)\| \\
&\leq |\lambda|\|\mathbb{M}(\phi_0)^{-1}\| \max_{0\leq k\leq m} \left|a_{ki}(\phi_1) - a_{ki}(\phi_0)\right| \\
&\leq QL\|\mathbb{M}(\phi_0)^{-1}\|\|\phi_1 - \phi_0\| \leq \delta_0 < 1,
\end{aligned}$$

it follows that $\mathbb{M}(\phi_1)^{-1}$ exists and $\|\mathbb{M}(\phi_1)^{-1}\| \leq \frac{\|\mathbb{M}(\phi_0)^{-1}\|}{1-\delta_0}$.

Now, we suppose that there exists $\mathbb{M}(\phi_k)^{-1}$ and $\|\mathbb{M}(\phi_k)^{-1}\| \leq \frac{\|\mathbb{M}(\phi_{k-1})^{-1}\|}{1-\delta_{k-1}}$, for all $k = 1, 2, \ldots, n$. Thus, by mathematical induction on n, we have

$$\begin{aligned}
\|I - \mathbb{M}(\phi_n)^{-1}\mathbb{M}(\phi_{n+1})\| &\leq \|\mathbb{M}(\phi_n)^{-1}\|\|\mathbb{M}(\phi_n) - \mathbb{M}(\phi_{n+1})\| \\
&\leq QL\|\mathbb{M}(\phi_n)^{-1}\|\|\phi_{n+1} - \phi_n\| \leq \delta_n < 1,
\end{aligned}$$

since $\{\delta_n\}$ is a decreasing function. As a consequence, there exists $\mathbb{M}(\phi_{n+1})^{-1}$. $\square$

Therefore, the algorithm of the Newton method defined previously is well defined. Moreover, provided that the conditions of Theorem 2 are satisfied, the Newton sequence (3.2) converges to a solution $\widetilde{\phi}$ of the equation $\mathcal{G}(\phi) = 0$. Note that the integrals that appear in the second step of the algorithm are approximations obtained with the Gauss numerical integration formula.

6 Examples

In this section, we present two examples that illustrate the previous study. In the first, the kernel $\mathcal{K}$ of (1.1) is non-separable and is non-differentiable in either of the two components, so we cannot approximate the kernel by a Taylor polynomial and then use the technique presented in this work. In the second, the kernel $\mathcal{K}$ of the integral equation (1.1) is non-separable but sufficiently differentiable into any of its two components. Then, we see that even in this situation our technique is competitive with respect to the application of the technique based on Taylor's polynomial.

Example 1. Consider the nonlinear integral equation

$$\phi(x) = f(x) + \frac{1}{2}\int_0^1 |x-t| \left(\frac{\phi(t)}{5}\right)^2 dt, \quad x \in [0,1],$$

with $f(x) = \frac{1}{600}\left(-2x^4 + 604x - 3\right)$, such that $\phi^*(x) = x$ is an exact solution.

As the kernel of the integral equation $\mathcal{K}(x,t) = |x-t|$ is non-separable and non-differentiable in any of its two components, we cannot approximate it by a Taylor series (something that is usually common when the kernel of the integral equation is separable [6]). But we observe that the kernel is Lipschitz continuous in the variable t, since $\|\mathcal{K}(x,t) - \mathcal{K}(x,s)\| \leq |t-s|$, for $s,t,x \in [0,1]$. Then, once $x \in [0,1]$ is fixed, the polynomial p_m converges to the kernel $\mathcal{K}$, so that there exists $m \in \mathbb{N}$ such that $\max_{t\in[0,1]}\left\{|\mathcal{K}(x,t) - p_m(x,t)|\right\}$ is as small as we want. If we use, for example, the polynomial (2.1) with $m = 5$, then the polynomial is not a good approximation of the kernel $\mathcal{K}(x,t) = |x-t|$, as we can see in Figure 1, so we increase the value of m up to $m = 30$ to obtain a good approximation of $\mathcal{K}$, as we can see in Figure 2. Hence, we contemplate the integral equation

$$\phi(x) = \frac{1}{600}\left(-2x^4 + 604x - 3\right) + \frac{1}{50}\int_0^1 p_{30}(x,t)\phi(t)^2\, dt, \quad x \in [0,1]. \quad (6.1)$$

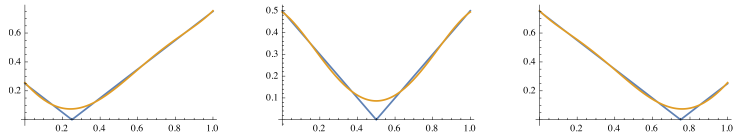

Figure 1. $\mathcal{K}(x,t)$ and $p_5(x,t)$ for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively.

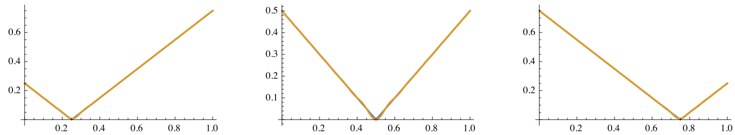

Figure 2. $\mathcal{K}(x,t)$ and $p_{30}(x,t)$ for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively.

Observe that the solution $\phi^*(x)$ can be located previously in a domain Ω, since it satisfies

$$\|\phi^*(x)\| - \frac{1}{600}\| - 2x^4 + 604x - 3\| - \frac{1}{50}\max_{0\le x,t\le 1}\left\{\int_0^1 |x-t|dt\right\}\|\phi^*(t)\|^2$$
$$\le \|\phi^*(x)\| - (0.9983\ldots) - (0.1)\|\phi^*(t)\|^2 \le 0,$$

which holds provided that $\|\phi^*(x)\| \le 1.0085\ldots$ As a consequence, the ball $\Omega = B(0, r^*)$, with $r^* = 1.0085\ldots$, contains $\phi^*(x)$.

The convergence of the Newton method is guaranteed from Theorem 2. As $L = 2$ and $Q = 0.02$, we can choose a starting function $\phi_0(x)$ such that $\|\mathcal{H}'(\phi_0)\| \le \frac{1}{Q} = 50$. For this, we choose, as it is usually done, the starting function $\phi_0(x) = f(x)$ (see [1, 2]). In this case, $\beta = 1.0415\ldots$, $\eta = 0.0207\ldots$ and $\delta_0 = QL\beta\eta = 0.0008\ldots < \frac{1}{2}$, so that the conditions of Theorem 2 are satisfied. Thus, we can then apply the Newton method from $\phi_0(x) = f(x)$ to approximate a solution $\widetilde{\phi}(x)$ of the integral equation (6.1).

Table 1. Error bounds from Example 1 and using the Newton method.

n	$\|\phi^*(x) - \phi_n(x)\|$
0	0.005
1	$6.4893\ldots \times 10^{-7}$
2	$9.5419\ldots \times 10^{-10}$

To obtain an approximation of the solution ϕ^*, we consider, for example, the stop criterion $\|\phi^*(x) - \phi_n(x)\| \le 10^{-15}$ for the Newton sequence. In Table 1, we show the errors obtained $\|\phi^*(x) - \phi_n(x)\|$, where $\phi_n(x)$ are the approximations given by the Newton method.

Finally, if we use a k-step Newton's method [8] with $k > 1$, instead of the modified of Newton method (1-step Newton's method), we can gain speed of convergence reducing the operational cost of the Newton method.

Example 2. Consider the following nonlinear integral equation

$$\phi(x) = f(x) + \frac{1}{70}\int_0^1 (x+2)\,e^{xt}\phi(t)^2\,dt, \quad x \in [0,1], \tag{6.2}$$

with $f(x) = \frac{1}{70}\left(1 + (70 - e^2)e^x\right)$, such that $\phi^*(x) = e^x$ is an exact solution.

Observe that the kernel of the integral equation is $\mathcal{K}(x,t) = (x+2)\,\mathrm{e}^{xt}$ is non-separable. Thus, we use the polynomial (2.1) to approximate it and consider the integral equation

$$\phi(x) = \frac{1}{70}\Big(1 + (70-\mathrm{e}^2)\mathrm{e}^x\Big) + \frac{1}{70}\int_0^1 p_m(x,t)\phi(t)^2\,dt, \quad x \in [0,1],$$

whose kernel is the polynomial p_m, which is separable.

Observe that $\phi^*(x)$ can be located previously in a domain Ω, since it satisfies

$$\|\phi^*(x)\| - \frac{1}{70}\|1+(70-\mathrm{e}^2)\mathrm{e}^x\| - \frac{1}{70}\max_{0\le x,t\le 1}\left\{\left|\int_0^1 (x+2)\,\mathrm{e}^{xt}dt\right|\right\}\|\phi^*(t)\|^2$$
$$\le \|\phi^*(x)\| - (2.4456\ldots) - \frac{3}{70}(\mathrm{e}-1)\|\phi^*(t)\|^2 \le 0,$$

which holds provided that $\|\phi^*(x)\| \le 3.1994\ldots$, so that the domain $\Omega = B(0,r^*)$, with $r^* = 3.1994\ldots$, contains $\phi^*(x)$.

Table 2. Error bounds based on Chebyshev's polynomial.

n	$\|\phi^*(x) - \phi_n(x)\|$
0	$1.7182\ldots$
1	$8.0116\ldots \times 10^{-2}$
2	$2.6846\ldots \times 10^{-4}$
3	$4.1782\ldots \times 10^{-9}$

Consider $m = 5$. The convergence of the Newton method is guaranteed from Theorem 2 if the conditions of the theorem are satisfied. Note that $L = 2$ and $Q = 0.1164\ldots$, so that we have to choose a starting function $\phi_0(x)$ such that $\|\mathcal{H}'(\phi_0)\| \le \frac{1}{Q} = 8.5838\ldots$ We choose $\phi_0(x) = 1$. Hence, $\beta = 1.3037\ldots$, $\eta = 0.1518\ldots$ and $\delta_0 = QL\beta\eta = 0.0461\ldots < \frac{1}{2}$, so that the hypotheses of Theorem 2 holds. So, we can then apply the Newton method from $\phi_0(x) = 1$ to approximate a solution $\phi^*(x)$ of the integral equation (6.2). In Table 2, we see the errors obtained $\|\phi^*(x) - \phi_n(x)\|$, where $\phi_n(x)$ are the approximations given by the Newton method.

Finally, we compare the results obtained with those obtained if the kernel of the integral equation (6.2) is approximated by Taylor's polynomial with $m+1 = 6$ terms, which are given in Table 3. From Tables 2 and 3, we observe that both polynomials have a similar behavior, although Chebyshev's polynomial behaves a little better in this case.

Table 3. Error bounds based on Taylor's polynomial.

n	$\|\phi^*(x) - \phi_n(x)\|$
0	$1.7182\ldots$
1	$8.0153\ldots \times 10^{-2}$
2	$3.3040\ldots \times 10^{-4}$
3	$6.2072\ldots \times 10^{-5}$

7 Conclusions

The Newton method is the most used iterative method to solve a nonlinear equation. The main problem of this method is the calculation of the inverse involved in its algorithm, specially when we work in spaces of infinite dimension. Thus, if we want to solve an integral equation, we have to guarantee the existence of the inverse at each step of the method, which is possible when the kernel is sufficiently differentiable into any of the components by approximating the kernel by a Taylor's polynomial. If the kernel is not differentiable in any of the components is studied in this work and we propose to approximate the kernel of a Fredholm integral equation of the second kind by a Chebyshev interpolation polynomial that lead to a separable kernel, so that we can explicitly calculate the inverse that appears in each step of the Newton method and then approximate a solution of the integral equation. Next, in future, we intend to extend and adapt this idea to the study of Volterra's integral equations.

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