ISSN: 1392-6292/ eISSN: 1648-3510

VILNIUS TECH Vilnius Gediminas Technical University

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 1
Pages 97–108
https://doi.org/10.3846/mma.2025.20817

Discrete universality theorem for Matsumoto zeta-functions and nontrivial zeros of the Riemann zeta-function

Keita Nakai

Graduate school of Mathematics, Nagoya University, Chikusa-Ku, 464-8602 Nagoya, Japan

Article History:
- received January 18, 2024
- revised June 6, 2024
- accepted August 22, 2024

Abstract. In 2017, Garunkštis, Laurinčikas and Macaitienė proved the discrete universality theorem for the Riemann zeta-function shifted by imaginary parts of nontrivial zeros of the Riemann zeta-function. This discrete universality has been extended to various zeta-functions and L-functions. In this paper, we generalize this discrete universality for Matsumoto zeta-functions.

Keywords: Matsumoto zeta-function; universality; nontrivial zeros.

AMS Subject Classification: 11M41.

Corresponding author. E-mail: m21029d@math.nagoya-u.ac.jp

1 Introduction

Let $s = \sigma + it$ be a complex variable. The Riemann zeta-function $\zeta(s)$ is defined by the infinite series $\sum_{n=1}^{\infty} n^{-s}$ in the $\sigma > 1$, and can be continued meromorphically to the whole plane $\mathbb{C}$. Let $K(r)$ be a disc with centre $3/4$ and radius r. In 1975, Voronin [23] proved that for any non-vanishing continuous function f and any $\varepsilon > 0$, there exists a positive τ for which

$$\sup_{s \in K(r)} |\zeta(s + i\tau) - f(s)| < \varepsilon$$

holds for $0 < r < 1/4$. This approximation theorem called the universality theorem. From Voronin's proof, the set of such τ has a positive density. Furthermore we can replace $K(r)$ by more general sets. The modern statement of universality theorem is as follow.

Theorem 1 [Voronin's universality theorem]. *Let $\mathcal{K}$ be a compact set in the strip $1/2 < \sigma < 1$ with connected complement, and let $f(s)$ be a non-vanishing continuous function on $\mathcal{K}$ that is analytic in the interior of $\mathcal{K}$. Then,* *for any* $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0,T] : \sup_{s\in\mathcal{K}} |\zeta(s+i\tau) - f(s)| < \varepsilon\Big\} > 0,$$

where meas *denotes the 1-dimensional Lebesgue measure.*

In this universality, the shift τ can take arbitrary non-negative real values continuously. If the shift can take certain values discretely and the universality holds by this shift, then we call it a discrete universality. First Reich [20] proved the discrete universality for the Dedekind zeta-function, and many mathematicians extended and generalized his result. See e.g., a survey paper [17] for the recent studies.

Let $0 < \gamma_1 \leq \gamma_2 \leq \dots$ be imaginary parts of nontrivial zeros of the Riemann zeta-function. Montgomery [19] conjectured the asymptotic relation

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ \frac{2\pi\alpha_1}{\log T}\leq \gamma-\gamma' \leq \frac{2\pi\alpha_2}{\log T}}} 1 \sim \left(\int_{\alpha_1}^{\alpha_2}\left(1-\left(\frac{\sin \pi u}{\pi u}\right)^2\right) du + \delta(\alpha_1,\alpha_2)\right)\frac{T}{2\pi}\log T$$

as $T\to\infty$ for $\alpha_1 < \alpha_2$, where $\delta(\alpha_1,\alpha_2) = 1$ if $0\in[\alpha_1,\alpha_2]$ and $\delta(\alpha_1,\alpha_2) = 0$ otherwise. We consider the weak Montgomery conjecture:

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ |\gamma-\gamma'|<c/\log T}} 1 \ll T\log T \tag{1.1}$$

as $T\to\infty$ with a certain constant $c > 0$. Under this conjecture, the following discrete universality for the Riemann zeta-function holds.

Theorem 2 [Garunkštis, Laurinčikas and Macaitienė [5]]. *Let* $\mathcal{K}$ *be a compact set in the strip* $1/2 < \sigma < 1$ *with connected complement, let* $f(s)$ *be a non-vanishing continuous function on* $\mathcal{K}$ *that is analytic in the interior of* $\mathcal{K}$ *and assume (1.1). Then, for any* $\varepsilon > 0$ *and* $h > 0$,

$$\liminf_{N\to\infty} \frac{1}{N}\#\Big\{1\leq k\leq N : \sup_{s\in\mathcal{K}} |\zeta(s+ih\gamma_k) - f(s)| < \varepsilon\Big\} > 0,$$

where $\#A$ *denotes the cardinality of a set* $A \subset \mathbb{N}$.

This universality theorem has been extended to other zeta-functions and L-functions in [2, 3, 6, 10, 11, 15]. In this paper, we prove this universality for the class of Matsumoto zeta-functions.

The notion of Matsumoto zeta-function $\varphi(s)$ is introduced by Matsumoto [16] and defined by

$$\varphi(s) = \prod_{n=1}^{\infty}\prod_{j=1}^{g(n)}(1 - a_n^{(j)}p_n^{-f(j,n)s})^{-1},$$

where $g(n) \in \mathbb{N}$, $f(j,n) \in \mathbb{N}$, $a_n^{(j)} \in \mathbb{C}$, and p_n is the nth prime number. Assuming the conditions

$$g(n) \le c_1 p_n^{\alpha},\ |a_n^{(j)}| \le p_n^{\beta} \tag{1.2}$$

with nonnegative constants α, β and a positive constant c_1, we have

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

for $\sigma > \alpha+\beta+1$. Furthermore, $b_n \ll n^{\alpha+\beta+\varepsilon}$ for any $\varepsilon > 0$ if all prime factors of n are large (see [7, Appendix]).

In this paper, we consider Matsumoto zeta-functions satisfying following assumptions.

(i) The condition (1.2).

(ii) There exists $\alpha+\beta+1/2 \le \rho < \alpha+\beta+1$ such that the function $\varphi(s)$ is meromorphic in the half plane $\sigma \ge \rho$, all poles in this region are included in a compact set, and there is no pole on the line $\sigma = \rho$.

(iii) There exists a positive constant c_2 such that $\varphi(\sigma+it) \ll |t|^{c_2}$ as $|t| \to \infty$ for $\sigma > \rho$.

(iv) For $\rho \le \sigma < \min\{\mathrm{Re}(z) : z \text{ is a pole of } \varphi\}$, we have

$$\int_{-T}^{T} |\varphi(\sigma+it)|^2\, dt \ll T.$$

(v) There exists a positive κ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p_n \le x} |\sum_{\substack{j=1 \\ f(j,n)=1}}^{g(n)} a_n^{(j)}|^2 p_n^{-2(\alpha+\beta)} = \kappa,$$

where $\pi(x)$ is the prime counting function.

Let $D_\rho = \{s \in \mathbb{C} : \rho < \sigma < \alpha+\beta+1\}$. Now we state the main theorem of this paper.

Theorem 3. *Let φ be a Matsumoto zeta-function satisfying (i)–(v). Let $\mathcal{K}$ be a compact set in D_ρ with connected complement, let $f(s)$ be a non-vanishing continuous function on $\mathcal{K}$ that is analytic in the interior of $\mathcal{K}$ and assume (1.1). Then, for any $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ N \le k \le 2N : \sup_{s\in\mathcal{K}} |\varphi(s+ih\gamma_k) - f(s)| < \varepsilon \right\} > 0.$$

We note that the class of Matsumoto zeta-functions satisfying (i)–(v) does not coinside with the Selberg class. There are difference points between Matsumoto zeta-functions and Selberg class. One example is that Matsumoto zeta functions can have poles other than $s = 1$, but L-functions in the Selberg class can have pole at $s = 1$ only.

Sourmelidis, Srichan and Steuding [21] proved similar universality for the Riemann zeta-function unconditionally. Their statement holds for the wider context of α-points of L-functions from the Selberg class. However, we have to take a subsequence of α-points of L-functions from the Selberg class in their result. Using their results, we have the following theorem without (1.1).

Theorem 4. *Let $\mathcal{K}$ and f be same as Theorem 3. Let $\mathcal{L}$ be a non-constant L-function in the Selberg class. Then, there exists a subsequence of α-points $(\rho_{\alpha,n_k})_{k\in\mathbb{N}}$ of $\mathcal{L}(s)$ such that for any $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{ N \le k \le 2N : \sup_{s\in\mathcal{K}} |\varphi(s + i\gamma_{\alpha,n_k}) - f(s)| < \varepsilon \Big\} > 0$$

holds, where $\gamma_{\alpha,n_k} = \mathrm{Im}\,(\rho_{\alpha,n_k})$.

Remark 1. In Theorem 4, we have to consider a subsequence of α-points of L-functions from the Selberg class same as [21, Theorem 5]. This reason comes from the fact that without (1.1) we can take a subsequence of α-points of L-functions from the Selberg class such that it is uniformly distributed in mod 1 and it can be approximated by certain values. However, it is difficult to compute such subsequence explicitly.

2 Preliminaries

We fix a compact subset $\mathcal{K}$ satisfying the assumptions of Theorem 3. We define $\rho < \sigma_0 < \min_{s\in\mathcal{K}} \mathrm{Re}\,(s)$ as all poles are contained in $\sigma > \sigma_0$. Then, we fix $\sigma_1,\ \sigma_2$ such that

$$\rho < \sigma_0 < \sigma_1 < \min_{s\in\mathcal{K}} \mathrm{Re}\,(s),\ \max_{s\in\mathcal{K}} \mathrm{Re}\,(s) < \sigma_2 < \alpha + \beta + 1.$$

Then, we define the rectangle region $\mathcal{R}$ by

$$\mathcal{R} = (\sigma_1,\ \sigma_2) \times i\Big(\min_{s\in\mathcal{K}} \mathrm{Im}\,(s) - 1/2,\ \max_{s\in\mathcal{K}} \mathrm{Im}\,(s) + 1/2 \Big). \tag{2.1}$$

Let $\mathcal{H}(\mathcal{R})$ be the set of all holomorphic functions on $\mathcal{R}$.

We write $\mathcal{B}(T)$ for the Borel set of T which is a topological space. Let $S^1 = \{s \in \mathbb{C} : |s| = 1\}$. For any prime p, we put $S_p = S^1$ and $\Omega = \prod_p S_p$. Then, there exists the probability Haar measure $\mathbf{m}$ on $(\Omega, \mathcal{B}(\Omega))$. Then $\mathbf{m}$ is written by $\mathbf{m} = \otimes_p \mathbf{m}_p$, where $\mathbf{m}_p$ is the probability Haar measure on $(S_p, \mathcal{B}(S_p))$.

Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space S_p. $\{\omega(p) :$ p prime$\}$ is a sequence of independent complex-valued random elements defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{m})$.

For $\omega \in \Omega$, we put $\omega(1) := 1$, $\omega(n) := \prod_p \omega(p)^{\nu(n;p)}$, where $\nu(n;p)$ is the exponent of the prime p in the prime factorization of n. Here, we define $\mathcal{H}(\mathcal{R})$-valued random elements

$$\varphi(s,\omega) := \prod_{n=1}^{\infty} \prod_{j=1}^{g(n)} (1 - a_m^{(j)} \omega(p)^{f(j,n)} p_n^{-f(j,n)s})^{-1} = \sum_{n=1}^{\infty} \frac{b_n \omega(n)}{n^s}.$$

We define probability measures on $(\mathcal{H}(\mathcal{R}), \mathcal{B}(\mathcal{H}(\mathcal{R})))$ by

$$\begin{aligned} P_N(A) =& \frac{1}{N+1} \# \left\{ N \le k \le 2N : \varphi(s + ih\gamma_k) \in A \right\}, \\ P(A) =& \mathbf{m} \left\{ \omega \in \Omega : \varphi(s,\omega) \in A \right\} \end{aligned}$$

for $A \in \mathcal{B}(\mathcal{H}(\mathcal{R}))$.

3 A limit theorem

This section is in the principle of Bagchi [1]. We can confirm Bagchi's method at Laurinčikas's book [12], Steuding's book [22] or Kowalski's book [9]. However, the way of taking φ_X (cf. after Lemma 1) based on Kowalski's book differs from Bagchi's original way. Certainly, Bagchi's original way is valid since the previous studies [2,3,6,10,11,15] are based on Bagchi's original way. However, this section and the way of taking φ_X are based on Kowalski's book.

Lemma 1. *Let $\psi : [0,\infty) \to \mathbb{C}$ be smooth and assume that ψ and all its derivatives decay faster than any polynomial at infinity, and let*

$$\hat{\psi}(s) = \int_0^{\infty} \psi(x) x^{s-1}\, dx$$

be the Mellin transform of ψ on $\mathrm{Re}(s) > 0$.

(1) The Mellin transform $\hat{\psi}$ extends to a meromorphic function on $\mathrm{Re}(s) > -1$, with at most a simple pole at $s = 0$ with residue $\psi(0)$.

(2) For any real numbers $-1 < A < B$, the Mellin transform has rapid decay in the strip $A \le \sigma \le B$, in the sense that for any integer $k \ge 1$, there exists a constant $C = C(k, A, B) \ge 0$ such that

$$|\hat{\psi}(\sigma + it)| \le C(1 + |t|)^{-k}$$

for all $A \le \sigma \le B$ and $|t| \ge 1$.

(3) For any $\sigma > 0$ and any $x \ge 0$, we have the Mellin inversion formula

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\psi}(s) x^{-s}\, ds.$$

Proof. See [9, Proposition A.3.1]. □

Now let

$$\psi_0(t) = e^{-\frac{1}{t}} I_{(0,\infty)}(t),$$

where $I_{(0,\infty)}$ is the indicator function on $(0,\infty)$. For $R > 1$ fixed, we define

$$\psi(x) = \frac{\psi_0(R^2 - x^2)}{\psi_0(R^2 - x^2) + \psi_0(|x|^2 - 1)}.$$

Then, $\psi(x)$ is a a real-valued smooth function on $[0,\infty)$ with compact support satisfying $\psi(x) = 1$ for $0 \le x \le 1$ and $0 \le \psi(x) \le 1$. Therefore, $\psi(x)$ satisfies assumptions of Lemma 1. Furthermore, we have

$$\hat{\psi}^{(k)}(\sigma + it) \ll_k (1 + |t|)^{-k}.$$

We put

$$\varphi_X(s) = \sum_{n=1}^{\infty} \frac{b_n \psi(n/X)}{n^s}, \quad \varphi_X(s,\omega) = \sum_{n=1}^{\infty} \frac{b_n \omega(n) \psi(n/X)}{n^s}$$

for $X \ge 2$.

Lemma 2. *For all compact set $C \subset \mathcal{R}$*

$$\lim_{X\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=N}^{2N} \sup_{s\in C} |\varphi(s + ih\gamma_k) - \varphi_X(s + ih\gamma_k)| = 0.$$

Proof.

From Lemma 1 (3) and definition of $\varphi_X(s)$, we see that

$$\varphi_X(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi(s+w)\hat{\psi}(w) X^w \, dw$$

for $c > \alpha + \beta + 1$. We write $z_1, \ldots, z_M$ for the poles of φ contained in $\overline{D}_\rho$ and $n_1, \ldots, n_M$ for its orders. Let $\delta(z)$ be a positive number satisfying $\mathrm{Re}(z) - \delta(z) = \sigma_0$ for $\mathrm{Re}(z) > \sigma_0$. If $z \ne z_j$ for $1 \le j \le M$ and $\mathrm{Re}(z) > \sigma_0$, then by the residue theorem, we have

$$\begin{aligned}\varphi(z) - \varphi_X(z) = &- \frac{1}{2\pi i} \int_{-\delta(z)-i\infty}^{-\delta(z)+i\infty} \varphi(z+w)\hat{\psi}(w) X^w \, dw \\ &- \sum_{j=1}^{M} \mathrm{Res}_{w=z_j - z} \varphi(z+w)\hat{\psi}(w) X^w.\end{aligned}$$

Since $\mathrm{Res}_{w=z_j-z}\varphi(z+w)\hat{\psi}(w)X^w$ can be represented by the linear form of $\hat{\psi}^{(l)}(z_j - z)(\log X)^{n_j - l} X^{z_j - z}$, we have

$$\mathrm{Res}_{w=z_j-z}\varphi(z+w)\hat{\psi}(w)X^w \ll_{n_j} (\log X)^{n_j} X^{\frac{1}{2}} (1 + |\mathrm{Im}(z - z_j)|)^{-1}. \quad (3.1)$$

Let N be sufficiently large. Then, $\varphi(s+ih\gamma_k)-\varphi_X(s+ih\gamma_k)$ is holomorphic on $\overline{\mathcal{R}}$ for $N\le k\le 2N$. Therefore, we have

$$\begin{aligned}
&\sum_{k=N}^{2N}\sup_{s\in C}|\varphi(s+ih\gamma_k)-\varphi_X(s+ih\gamma_k)|\\
&=\frac{1}{2\pi}\sum_{k=N}^{2N}\sup_{s\in C}\left|\int_{\partial\mathcal{R}}\frac{\varphi(z+ih\gamma_k)-\varphi_X(z+ih\gamma_k)}{z-s}\,dz\right|\\
&\le\frac{1}{2\pi\mathrm{dist}(C,\partial\mathcal{R})}\int_{\partial\mathcal{R}}\sum_{k=N}^{2N}|\varphi(z+ih\gamma_k)-\varphi_X(z+ih\gamma_k)||dz|\\
&\le\frac{1}{4\pi^2\mathrm{dist}(C,\partial\mathcal{R})}\int_{\partial\mathcal{R}}\sum_{k=N}^{2N}\int_{-\delta(z)-i\infty}^{-\delta(z)+i\infty}|\varphi(z+w+ih\gamma_k)||\hat{\psi}(w)X^w|\,|dw||dz|\\
&\quad+\frac{|\partial\mathcal{R}|}{2\pi\mathrm{dist}(C,\partial\mathcal{R})}\sup_{z\in\partial\mathcal{R}}\sum_{k=N}^{2N}\sum_{j=1}^{M}\mathrm{Res}_{w=z_j-z-ih\gamma_k}\varphi(z+w+ih\gamma_k)\hat{\psi}(w)X^w\\
&\le\frac{|\partial\mathcal{R}|}{4\pi^2\mathrm{dist}(C,\partial\mathcal{R})}\sup_{z\in\partial\mathcal{R}}X^{-\delta(z)}\\
&\quad\times\int_{-\infty}^{\infty}\sum_{k=N}^{2N}|\varphi(\mathrm{Re}(z)-\delta(z)+i\tau+ih\gamma_k)||\hat{\psi}(-\delta+i\tau)|\,d\tau\\
&\quad+\frac{|\partial\mathcal{R}|}{2\pi\mathrm{dist}(C,\partial\mathcal{R})}\sup_{z\in\partial\mathcal{R}}\sum_{k=N}^{2N}\sum_{j=1}^{M}\mathrm{Res}_{w=z_j-z-ih\gamma_k}\varphi(z+w+ih\gamma_k)\hat{\psi}(w)X^w,
\end{aligned}$$

where $\mathrm{dist}(C,\partial\mathcal{R})$ is the minimal distance between C and $\partial\mathcal{R}$, and $|\partial\mathcal{R}|$ is the length of $\partial\mathcal{R}$.

We consider the first term. Using (1.1), assumption (iv) and the Gallagher Lemma on the discrete mean (see [18, Lemma 1.4]), we have

$$\frac{1}{N+1}\sum_{k=N}^{2N}|\varphi(\mathrm{Re}(z)-\delta(z)+i\tau+ih\gamma_k)|\ll 1+|\tau|,$$

(cf. [5, Lemma 2.7]). Thus, we obtain

$$\frac{1}{N+1}\sup_{z\in\partial\mathcal{R}}\int_{-\infty}^{\infty}\sum_{k=N}^{2N}|\varphi(\mathrm{Re}(z)-\delta(z)+i\tau+ih\gamma_k)||\hat{\psi}(-\delta(z)+i\tau)|\,d\tau\ll 1.$$

We consider the second term. It is known that $\gamma_k\sim 2\pi k/\log k$ as $k\to\infty$, so we have

$$\gamma_k\gg\frac{k}{\log k}.$$

By Lemma 1 and (3.1), we have

$$\begin{aligned}
&\sup_{z\in\partial\mathcal{R}}\sum_{k=N}^{2N}\sum_{j=1}^{M}\mathrm{Res}_{w=z_j-z-ih\gamma_k}\varphi(z+w+ih\gamma_k)\hat{\psi}(w)X^w\\
&\ll_{n_j}\sup_{z\in\partial\mathcal{R}}\sum_{k=N}^{2N}\sum_{j=1}^{M}(\log X)^{n_j}X^{\frac{1}{2}}(1+|\mathrm{Im}(z_j)-\mathrm{Im}(z)-h\gamma_k|)^{-1}\\
&\ll_{M,z_j,\mathcal{R}} X^{\frac{1}{2}}\sup_{1\le j\le M}(\log X)^{n_j}\sum_{k=N}^{2N}\frac{\log k}{k}\ll_{n_j,\varepsilon} X^{\frac{1}{2}+\varepsilon}(\log N)
\end{aligned}$$

for $\varepsilon>0$. Since $\delta(z)\ge\sigma_1-\sigma_0>0$ for all $z\in\partial\mathcal{R}$, we conclude

$$\begin{aligned}
&\lim_{X\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=N}^{2N}\sup_{s\in C}|\varphi(s+ih\gamma_k)-\varphi_X(s+ih\gamma_k)|\\
&\ll\lim_{X\to\infty}\limsup_{N\to\infty}(X^{-(\sigma_1-\sigma_0)}+X^{\frac{1}{2}+\varepsilon}(\log N)N^{-1})=0.
\end{aligned}$$

□

Lemma 3. *The following statements hold.*

(i) The product $\prod_{n=1}^{\infty}\prod_{j=1}^{g(n)}(1-a_m^{(j)}\omega(p)^{f(j,n)}p_n^{-f(j,n)s})^{-1}$ *and the series* $\sum_{n=1}^{\infty}b_n\omega(n)n^{-s}$ *are holomorphic on the domain* $\sigma>\alpha+\beta+1/2$ *for almost all* $\omega\in\Omega$.

(ii) For $\sigma>(\sigma_0+\sigma_1)/2$,

$$\mathbb{E}^{\mathbf{m}}[|\varphi(s,\omega)|]\ll_{\mathcal{K},\sigma_0}1+|t|$$

holds.

Proof. Applying the Kolmogorov theorem (see [12, Theorem 1.2.11]) and the convergence theorem relating with orthogonal random elements (see [12, Theorem 1.2.9]), we can prove (i).

We consider (ii). Let

$$S(u)=\sum_{n\le u}\frac{b_n\omega(n)}{n^{\sigma_0}}.$$

By the Cauchy–Schwartz inequality, there exists $M>0$ such that

$$\mathbb{E}^{\mathbf{m}}[|S_u|]\le\left(\mathbb{E}^{\mathbf{m}}[|S_u|^2]\right)^{\frac{1}{2}}=\left(\sum_{n\le u}|b_n|^2/n^{2\sigma_0}\right)^{\frac{1}{2}}<M.$$

By the definition of S_u, we have

$$\varphi(s,\omega)=\int_{1-}^{\infty}\frac{1}{u^{s-\sigma_0}}\,dS_u=(s-\sigma_0)\int_1^{\infty}\frac{S_u}{u^{s-\sigma_0-1}}\,du.$$

Thus, for $\mathrm{Re}(s) > (\sigma_0+\sigma_1)/2$, we have

$$\mathbb{E}^{\mathbf{m}}[|\varphi(s,\omega)|] \le |s-\sigma_0| \int_1^\infty \frac{\mathbb{E}^{\mathbf{m}}[|S_u|]}{u^{\sigma-\sigma_0-1}}\, du \le M \frac{|s-\rho|}{\sigma-\sigma_0} \ll_{\mathcal{K},\sigma_0} 1+|t|.$$

□

From this lemma, we see that

$$\varphi(s,\omega) = \prod_{n=1}^{\infty} \prod_{j=1}^{g(n)} (1-a_m^{(j)}\omega(p)^{f(j,n)} p_n^{-f(j,n)s})^{-1} = \sum_{n=1}^{\infty} \frac{b_n\omega(n)}{n^s}$$

holds for $\sigma > \rho$ for almost all $\omega \in \Omega$ in the sense of analytic continuation.

Lemma 4. *For all compact sets $C \subset \mathcal{R}$,*

$$\lim_{X\to\infty} \mathbb{E}^{\mathbf{m}}[\sup_{s\in C} |\varphi(s,\omega) - \varphi_X(s,\omega)|].$$

Proof. By Lemma 3 (i) we have

$$\sup_{s\in C} |\varphi(s,\omega)-\varphi_X(s,\omega)| \ll X^{-\delta} \int_{\partial\mathcal{R}} |dz| \int_{-\infty}^{\infty} |\varphi(z-\delta+i\tau,\omega)||\hat{\psi}(-\delta+i\tau)|\, d\tau,$$

where $\delta = (\sigma_1-\sigma_0)/4$ in the same way as Lemma 2. From Lemma 3 (ii), we have

$$\begin{aligned}&\mathbb{E}^{\mathbf{m}}[\sup_{s\in C} |\varphi(s,\omega)-\varphi_X(s,\omega)|]\\ &\ll X^{-\delta} \int_{\partial\mathcal{R}} |dz| \int_{-\infty}^{\infty} \mathbb{E}^{\mathbf{m}}[|\varphi(z-\delta+i\tau,\omega)|]|\hat{\psi}(-\delta+i\tau)|\, d\tau \ll X^{-\delta} \to 0\end{aligned}$$

as $X\to\infty$. □

We consider the discrete topology on $\mathbb{N}_{N\le n\le 2N} := \{n\in\mathbb{N} : N \le n \le 2N\}$. Then we define the probability measure on $(\mathbb{N}_{N\le n\le 2N}, \mathcal{B}(\mathbb{N}_{N\le n\le 2N}))$ by

$$\mathbb{P}_N(A) = \frac{1}{N+1}\#A,$$

for $A \in \mathcal{B}(\mathbb{N}_{N\le n\le 2N})$. Furthermore, let $\mathcal{P}_0$ be a finite set of prime numbers, and we define the probability measure on $(\prod_{p\in\mathcal{P}_0} S_p, \mathcal{B}(\prod_{p\in\mathcal{P}_0} S_p))$ by

$$Q_N^{\mathcal{P}_0}(A) = \frac{1}{N+1}\#\left\{N \le k \le 2N : (p^{ih\gamma_k})_{p\in\mathcal{P}_0} \in A\right\},$$

for $A \in \mathcal{B}(\prod_{p\in\mathcal{P}_0} S_p)$. Then, the next lemma holds.

Lemma 5. *The probability measure $Q_N^{\mathcal{P}_0}$ converges weakly to $\otimes_{p\in\mathcal{P}_0}\mathbf{m}_p$ as $N\to\infty$.*

Proof. We can prove this lemma in the same way as [5, Theorem 2.3]. □

Proposition 1. *The probability measure P_N converges weakly to P as $N \to \infty$.*

Proof. Using the Portmanteau theorem (see [8, Theorem 13.16]), Lemma 2, Lemma 4 and Lemma 5, we can prove this Proposition (cf. [4, Proposition 1]). □

4 Proof of main theorems

Let

$$S := \{f \in \mathcal{H}(\mathcal{R}) : \ f(s) \neq 0 \text{ or } f(s) \equiv 0\}.$$

Then, using assumption (v) and same method [13, Lemma 6], we see that the support of the measure P coincides with S (cf. [14, Lemma 6]).

Proof of Theorem 3. Let $\mathcal{K}$ be a compact set in D_ρ with connected complement, let f be a non-vanishing continuous function on $\mathcal{K}$ that is analytic in the interior of $\mathcal{K}$. We define $\mathcal{R}$ as (2.1). Fix $\varepsilon > 0$.

By the Mergelyan theorem, there exists a polynomial $G(s)$ such that

$$\sup_{s \in \mathcal{K}} |f(s) - \exp(G(s))| < \varepsilon/2$$

since f is non-vanishing on $\mathcal{K}$. Here we define an open set of $\mathcal{H}(\mathcal{R})$ by

$$\Phi(G) := \Big\{ g \in \mathcal{H}(\mathcal{R}) : \sup_{s \in \mathcal{K}} |g(s) - \exp(G(s))| < \varepsilon/2 \Big\}.$$

Applying the Portmanteau theorem, Proposition 1, and a property of the support of P, we have

$$\begin{aligned} &\liminf_{N \to \infty} \frac{1}{N+1} \# \Big\{ N \le k \le 2N : \sup_{s \in \mathcal{K}} |\varphi(s + ih\gamma_k) - \exp(G(s))| < \varepsilon/2 \Big\} \\ &= \liminf_{N \to \infty} P_N(\Phi(G)) \ge P(\Phi(G)) > 0. \end{aligned}$$

Now, the inequality

$$\sup_{s \in \mathcal{K}} |\varphi(s + ih\gamma_k) - f(s)| \le \sup_{s \in \mathcal{K}} |\varphi(s + ih\gamma_k) - \exp(G(s))| + \sup_{s \in \mathcal{K}} |\exp(G(s)) - f(s)|$$

holds. Thus, we obtain Theorem 3. □

Finally, we prove Theorem 4. We utilize the following theorem.

Theorem 5. *Let $\mathcal{L}$ be a non-constant L-function in the Selberg class, $b > 0$ be a real number and alpha a complex number. Then, there exists a subsequence of alpha-points $(\rho_{\alpha,n_k})_{k\in\mathbb{N}}$, of $\mathcal{L}(s)$, such that $\gamma_{\alpha,n_k} = bk + o(1)$, and the sequence $(a\gamma_{\alpha,n_{k^m}})_{k\in\mathbb{N}}$ is uniformly distributed mod 1 for every real number $a \notin b^{-1}\mathbb{Q}$ and every positive integer m.*

Proof. This is Corollary 1 in [21] and we can find this proof in [21]. □

Proof of Theorem 4. In Lemma 2 and Lemma 5, we replace γ_k by $\gamma_{\alpha,k}$. Using Theorem 5 and proceeding along the same line as in the proof of Lemma 2 and Lemma 5 respectively, we can prove analogue of Lemma 2 and Lemma 5 with γ_k replaced by $\gamma_{\alpha,k}$. Therefore we obtain Theorem 4. □

Acknowledgements

The author would like to thank referees and Professor Kohji Matsumoto for their helpful comments. The author is grateful to Professor Jörn Steuding for making me aware of theirs paper. The author also would like to thank Professor Kenta Endo to point out mistakes of Lemma 2. This work was financially supported by JST SPRING, Grant Number JPMJSP2125.

References

[1] B. Bagchi. *Statistical behaviour and universality properties of the Riemann zeta function and other allied Dirichlet series*. PhD thesis, Indian Statistical Institute-Kolkata, 1981.

[2] A. Balčiūnas, V. Franckevič, V. Garbaliauskienė, R. Macaitienė and A. Rimkevičienė. Universality of zeta-functions of cusp forms and non-trivial zeros of the Riemann zeta-function. *Math. Model. Anal.*, **26**(1):82–93, 2021. https://doi.org/10.3846/mma.2021.12447.

[3] A. Balčiūnas, V. Garbaliauskienė, J. Karaliūnaitė, R. Macaitienė, J. Petuškinaitė and A. Rimkevičienė. Joint discrete approximation of a pair of analytic functions by periodic zeta-functions. *Math. Model. Anal.*, **25**(1):71–87, 2020. https://doi.org/10.3846/mma.2020.10450.

[4] K. Endo. Universality theorem for the iterated integrals of the logarithm of the Riemann zeta-function. *Lith. Math. J.*, **62**:315–332, 2022. https://doi.org/10.1007/s10986-022-09568-7.

[5] R. Garunkštis, A. Laurinčikas and R. Macaitienė. Zeros of the Riemann zeta-function and its universality. *Acta Arith.*, **181**:127–142, 2017. https://doi.org/10.4064/aa8583-5-2017.

[6] R. Kačinskaitė. On discrete universality in the Selberg–Steuding class. *Sib. Math. J.*, **63**(2):277–285, 2022. https://doi.org/10.1134/S0037446622020069.

[7] R. Kačinskaitė and K. Matsumoto. Remarks on the mixed joint universality for a class of zeta functions. *Bull. Aust. Math. Soc.*, **95**(2):187–198, 2017. https://doi.org/10.1017/S0004972716000733.

[8] A. Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2020. https://doi.org/10.1007/978-3-030-56402-5.

[9] E. Kowalski. *An introduction to probabilistic number theory*, volume 192. Cambridge University Press, 2021. https://doi.org/10.1017/9781108888226.

[10] A. Laurinčikas. Non-trivial zeros of the Riemann zeta-function and joint universality theorems. *J. Math. Anal. Appl.*, **475**(1):385–402, 2019. https://doi.org/10.1016/j.jmaa.2019.02.047.

[11] A. Laurinčikas. Zeros of the Riemann zeta-function in the discrete universality of the Hurwitz zeta-function. *Stud. Sci. Math. Hung.*, **57**(2):147–164, 2020. https://doi.org/10.1556/012.2020.57.2.1460.

[12] A. Laurinčikas. *Limit Theorems for the Riemann Zeta-function*. Kluwer, 1996. https://doi.org/10.1007/978-94-017-2091-5.

[13] A. Laurinčikas. On the Matsumoto zeta-function. *Acta Arith.*, **84**:1–16, 1998. https://doi.org/10.4064/aa-84-1-1-16.

[14] A. Laurinčikas and K. Matsumoto. The universality of zeta-functions attached to certain cusp forms. *Acta Arith.*, **98**(4):345–359, 2002. https://doi.org/10.4064/aa98-4-2.

[15] A. Laurinčikas and J. Petuškinaitė. Universality of Dirichlet L-functions and non-trivial zeros of the Riemann zeta-function. *Sb. Math.*, **210**(12):1753–1773, 2019. https://doi.org/10.1070/SM9194.

[16] K. Matsumoto. Value-distribution of zeta-functions. *Lecture Notes in Math.*, **1434**:178–187, 1990. https://doi.org/10.1007/BFb0097134.

[17] K. Matsumoto. A survey on the theory of universality for zeta and L-functions. *Number Theory: Plowing and Starring Through High Wawe Forms, Proc. 7th China-Japan Semin.(Fukuoka 2013)*, **11**:95–144, 2015. https://doi.org/10.1142/9789814644938_0004.

[18] H.L. Montgomery. *Topics in Multiplicative Number Theory.* Springer, 1971.

[19] H.L. Montgomery. The pair correlation of zeros of the zeta function. In *Proc. Symp. Pure Math*, volume 24, pp. 181–193, 1973. https://doi.org/10.1007/BFb0060851.

[20] A. Reich. Werteverteilung von Zetafunktionen. *Arch. Math.*, **34**:440–451, 1980. https://doi.org/10.1007/BF01224983.

[21] A. Sourmelidis, T. Srichan and J. Steuding. On the vertical distribution of values of L-functions in the Selberg class. *Int. J. Number Theory*, **18**(2):277–302, 2022. https://doi.org/10.1142/S1793042122500191.

[22] J. Steuding. *Value-distribution of L-functions*, volume 1877. Springer, 2007. https://doi.org/10.5565/PUBLMAT_PJTN05_12.

[23] S.M. Voronin. Theorem on the "universality" of the Riemann zeta-function. *Izv. Akad. Nauk SSSR Ser. Mat.*, **39**(3):475–486, 1975. https://doi.org/10.1070/IM1975v009n03ABEH001485.