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The BKM Criterion to the 3D Double-Diffusive Magneto Convection Systems Involving Planar Components

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Abstract. In this paper, we investigate the BKM type blowup criterion applied to 3D double-diffusive magneto convection systems. Specifically, we demonstrate that a unique local strong solution does not experience blow-up at time T, given that $(\nabla_h\tilde{u}, \nabla_h\tilde{b}) \in L^2(0,T;\dot{B}^{-1}_{\infty,\infty})$. To prove this, we employ the logarithmic Sobolev inequality in the Besov spaces with negative indices and a well-known commutator estimate established by Kato and Ponce. This result is the further improvement and extension of the previous works by O (2021) and Wu (2023).

Keywords: double-diffusive convection systems, blowup criterion, commutator estimate, regularity.

AMS Subject Classification: 35Q35; 35B35; 76D03.

1 Introduction and main result

We are concerned with the following double-diffusive magneto convection systems in $(0,T)\times\mathbb{R}^3$:

$$\begin{cases} u_t-\Delta u+(u\cdot\nabla)u+\nabla p=(b\cdot\nabla)b+(\theta-s)\mathbf{k}, \\ b_t-\Delta b+(u\cdot\nabla)b=(b\cdot\nabla)u, \\ \theta_t-\Delta\theta+u\cdot\nabla\theta=u\cdot\mathbf{k}, \\ s_t-\Delta s+u\cdot\nabla s=u\cdot\mathbf{k}, \\ \nabla\cdot u=\nabla\cdot b=0 \\ u(0,x)=u_0(x), b(0,x)=b_0(x), \theta(0,x)=\theta_0(x), s(0,x)=s_0(x), \end{cases} \tag{1.1}$$

where $u=u(t,x)$, $b=b(t,x)$ and $p=p(t,x)$are unknown velocity, magnetic field and dynamic pressure of the fluid, respectively. $\theta=\theta(t,x)$, $s=s(t,x)$ and $\mathbf{k}$ are scalar quantities affecting the density of the fluid and vertical unit vector, respectively.

The behavior of double-diffusive convection is initiated by the interplay between two fluid constituents diffusing at varying speeds, and it holds significant prominence in diverse areas such as oceanography and numerous other scientific domains (see [7, 8, 13] for details). The system of equations (1.1) at $b\equiv\theta\equiv s\equiv 0$ degenerates to the Navier-Stokes equations (NSE for short) and the system of (1.1) at $\theta\equiv s\equiv 0$ degenerates to the magnetohydrodynamics (MHD) equations and the system (1.1) at $b\equiv 0$ degenerates to the double-diffusive convection system. Intensive studies have been conducted on all of those systems, specifically focusing on whether the given initial data is smooth enough for the solution to remain smooth or develop a singularity in finite time.

In a recent work by Wu [16], it has been shown that there exists a unique local strong solution to the problem (1.1) when given initial data u_0 in $H^1(\mathbb{R}^3)$. Furthermore, they have proven that the strong solution can be globally extended when the L^2 norm of the initial data is small.

The question of whether a local strong solution can be extended beyond T up to infinity remains open for the 3D NSE. Numerous studies have attempted to provide sufficient conditions to ensure this extension of the local strong solution(see [1, 10, 11] and references therein). One of the pioneering works is the Beale-Kato-Majda criterion. It was proved in [3] that a unique local strong solution u to Euler equations can be extended beyond T if the vorticity satisfies

$$\omega\in L^1(0,T;L^\infty). \tag{1.2}$$

It is well known that BKM criterion (1.2) also holds for the NSE. Recently, Guo, Kučera and Skalák [6] established the regularity criteria to the 3D NSE via two vorticity components ($\tilde{\omega}=(\omega_1,\omega_2,0)$). More precisely, they proved that if

$$\tilde{\omega}\in L^p(0,T;\dot{B}^{-3/q}_{\infty,\infty}),\ \text{for } q\in(3,\infty) \text{ and } 2/p+3/q=2,$$

or

$$\tilde{\omega}\in L^p(0,T;\dot{B}^{-3(1/q-1/\theta)}_{\theta,\infty}),\ \text{for } q\in(3/2,3],\ \theta\in[q,3q/(3-q)) \text{ and } 2/p+3/q=2,$$

then the weak solution is smooth. After that, the first author in [10] proved the following continuation criteria

$$\widetilde{\omega} \in L^2(0,T;BMO^{-1}). \tag{1.3}$$

Quite recently, the first author of the present paper further extended (1.3) to the largest critical space $\dot{B}^{-1}_{\infty,\infty}$ [11].

Let us briefly sum up the regularity criteria concerning the partial derivative of planar components. We denote by $\tilde{u} = (u_1, u_2, 0)$, $\tilde{b} = (b_1, b_2, 0)$ and $\nabla_h = (\partial_1, \partial_2, 0)$ the planar components vector of u, b and the partial derivative, respectively. Dong and Zhang [5] proved the BKM type criterion to the 3D NSE via the partial derivative of planar velocity $\nabla_h \tilde{u}$

$$\nabla_h \tilde{u} \in L^1(0,T;\dot{B}^0_{\infty,\infty}).$$

In papers [12, 14, 15], they established some global regularity and stability results to the 3D double-diffusive convection system. In particular, the author in [14] proved that if partial derivatives of the planar components of the velocity field (i.e., $\nabla_h \tilde{u}$) belong to the Besov space:

$$\nabla_h \tilde{u} \in L^{\frac{2}{2-r}}\left(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)\right) \quad \text{with} \quad 0 \le r < 1, \tag{1.4}$$

then the local solution (u, θ, s) can be extended smoothly beyond $t = T$. It is an open question for the case of end-point $r = 1$ in (1.4) at that time. In the case of the 3D double-diffusive magneto convection system, the second author of the present paper [16] proved the BKM type blow-up criterion involving the partial derivative of planar components. Namely, it proved that the local strong solution can be extended beyond T if

$$(\nabla_h \tilde{u}, \nabla_h \tilde{b}) \in L^2(0,T;BMO^{-1}). \tag{1.5}$$

Motivated by the works cited above, it is interesting to extend the criterion (1.5) to the larger space. In this article, we consider an sufficient condition, in terms of the partial derivative of planar components $(\nabla_h \tilde{u}, \nabla_h \tilde{b})$ in the largest critical space, that guarantee the global regularity of the 3D double-diffusive magneto convection system (1.1). Specifically, we have the following result.

Theorem 1. Let $(u_0, b_0, \theta_0, s_0) \in H^2$ and (u, b, θ, s) be a unique local strong solution to (1.1) with initial data $(u_0, b_0, \theta_0, s_0)$. If $(\tilde{u}, \tilde{b})$ satisfies the following condition

$$\int_0^T \left(\|\nabla_h \tilde{u}\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla_h \tilde{b}\|^2_{\dot{B}^{-1}_{\infty,\infty}} \right) dt < \infty, \tag{1.6}$$

then (u, b, θ, s) can be smoothly extended beyond time T.

Remark 1. One can see that the continuous embedding

$$\dot{H}^{1/2} \hookrightarrow L^3 \hookrightarrow \dot{B}^{-1+3/\theta}_{\theta,\infty} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}, \; 3 \le \theta < \infty \tag{1.7}$$

hold in $\mathbb{R}^3$. *In view of* (1.7), *criterion* (1.6) *can be viewed as a generalization of [10, 11, 14] and [16]. In particular, Theorem 1 solves the remaining problem in [14].*

2 Preliminaries

We will introduce the homogeneous Besov space, the BMO space and the BMO^{-1} space (see [2, 4]). Let $\mathcal{S}'$ be the space of tempered distributions. $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Let B be the ball $B = \{\xi \in \mathbb{R}^3 \mid |\xi| \le 4/3\}$ and $\mathcal{C}$ be the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^3 \mid 3/4 \le |\xi| \le 8/3\}$. Then, there exist radial smooth functions χ and φ with values in the interval $[0,1]$, and supports respectively in B and $\mathcal{C}$ such that

$$\begin{aligned}\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) =& 1, \quad \xi \in \mathbb{R}^3,\\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) =& 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.\end{aligned}$$

The *homogeneous dyadic blocks* $\dot{\Delta}_j$ and *homogeneous low-frequency cut-off operators* $\dot{S}_j$ are defined for all $u \in \mathcal{S}'$ and $j \in \mathbb{Z}$ by

$$\begin{aligned}\dot{\Delta}_j u =& \varphi(2^{-j}D)u = \mathcal{F}^{-1}\varphi_j * u,\\ \dot{S}_j u =& \chi(2^{-j}D)u = \mathcal{F}^{-1}\chi_j * u.\end{aligned}$$

We denote by $\mathcal{S}'_h$ the space of tempered distributions u such that

$$\lim_{j\to-\infty}\|\dot{S}_j u\|_\infty = 0.$$

Let $s \in \mathbb{R}$ and $p, q \in [1,\infty]$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined as follows:

$$\begin{aligned}\dot{B}^s_{p,q} =& \{u \in \mathcal{S}'_h \mid \|u\|_{\dot{B}^s_{p,q}} < \infty\},\\ \|u\|_{\dot{B}^s_{p,q}} =& \begin{cases}\left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\dot{\Delta}_j u\|_p^q\right)^{\frac{1}{q}}, & 1 \le p \le \infty,\ 1 \le q < \infty,\\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_p, & 1 \le p \le \infty,\ q = \infty.\end{cases}\end{aligned}$$

The space BMO consists of locally integrable functions f such that

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |f - f_B|dx < \infty \text{ with } f_B = \frac{1}{|B|}\int_B f dx,$$

where the the supremum is taken over all balls B in $\mathbb{R}^3$. The space BMO^{-1} is defined by

$$BMO^{-1} = \{f \in \mathcal{S}' \mid \text{There exists } g = (g_1, g_2, g_3) \in BMO \text{ such that } f = \sum_{i=1}^3 \partial_i g_i\}$$

with the norm

$$\|f\|_{BMO^{-1}} = \inf_{g\in BMO} \sum_{i=1}^3 \|g_i\|_{BMO}.$$

We recall some lemmas that will be used in the proof of our result.

Lemma 1. *[10] For any $f \in BMO^{-1}, g \in W_q^1$ and $h \in W_r^1$, it holds that*

$$\int_{\mathbb{R}^3} fgh\,dx \le C\|f\|_{BMO^{-1}}(\|g\|_q\|\nabla h\|_r + \|\nabla g\|_q\|h\|_r),$$

where $1 < q, r < \infty$, $1/q + 1/r = 1$.

Lemma 2. *[11] Let $s > 1/2$, then, there exists a constant $C > 0$ such that*

$$\|f\|_{\dot{B}^{-1}_{\infty,2}} \le C(1 + \|f\|_{\dot{B}^{-1}_{\infty,\infty}} \ln^{1/2}(e + \|f\|_{H^s}))$$

for every $f \in H^s(\mathbb{R}^3)$.

We will use the following commutator estimate due to Kato and Ponce.

Lemma 3. *[9] Let $1 < p < \infty$ and $s > 0$. Then, there exists an constant C such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_p \le C(\|\nabla f\|_{p_1}\|\Lambda^{s-1}g\|_{p_2} + \|\Lambda^s f\|_{p_3}\|g\|_{p_4}),$$

for $f \in \dot{W}^{1,p_1} \cap \dot{W}^{s,p_3}$, $g \in L^{p_4} \cap \dot{W}^{s-1,p_2}$ and $1 < p_2, p_3 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, where $\Lambda^s = (-\Delta)^{s/2}$.

3 Proof of the main result

In this section we prove Theorem 1.

Proof. The proof is based on the establishment of a priori estimate for (u, b, θ, s) that allows us to extend the smooth solution beyond time T. If (1.6) holds, for any small constant $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) \in (0, T)$ such that

$$\int_{T_0}^{T} \left(\|\nabla_h \widetilde{u}\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla_h \widetilde{b}\|^2_{\dot{B}^{-1}_{\infty,\infty}}\right) d\tau \le \epsilon.$$

For any $t \in (T_0, T)$, we denote

$$X(t) = \max_{\tau \in [T_0, t]} (\|u(\tau)\|^2_{H^2} + \|b(\tau)\|^2_{H^2}).$$

Note that $X(t)$ is nondecreasing. The proof is divided into two steps.

First, we show the basic energy estimate. Taking the L^2 inner product to systems $(1.1)_1$, $(1.1)_2$, $(1.1)_3$ and $(1.1)_4$ with u, b, θ and s, respectively, we obtain

$$\|(u, b, \theta, s)(t)\|_2^2 + 2\int_0^T \|\nabla(u, b, \theta, s)(\tau)\|_2^2\,d\tau \le C_0. \tag{3.1}$$

Taking the gradient operator to $(1.1)_1$ and $(1.1)_2$, and multiply the resulting equations by ∇u and ∇b, respectively. Integrating over whole space and summing up the resulting identities gives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_2^2+\|\nabla b\|_2^2)+(\|\Delta u\|_2^2+\|\Delta b\|_2^2)\\
&=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u dx+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla b)\cdot\nabla u dx+\int_{\mathbb{R}^3}\nabla(\theta-s)\mathbf{k}\cdot\nabla u dx\\
&\quad-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b dx\quad+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla u)\cdot\nabla b dx=\sum_{i=1}^{5}I_i.
\end{aligned}\tag{3.2}$$

As argued in [10, 16], it follows from Lemma 1 and Young's inequality that

$$\begin{aligned}
I_1+I_2+I_4+I_5\leq &C\left(\|\nabla_h\widetilde{u}\|_{BMO^{-1}}^2+\|\nabla_h\widetilde{b}\|_{BMO^{-1}}^2\right)\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right)\\
&+\frac{1}{4}\left(\|\Delta u\|_2^2+\|\Delta b\|_2^2\right).
\end{aligned}\tag{3.3}$$

For I_3, we have

$$I_3=\int_{\mathbb{R}^3}\nabla(\theta-s)\mathbf{k}\cdot\nabla u dx\leq\frac{1}{2}\left(\|\nabla u\|_2^2+\|\nabla\theta\|_2^2+\|\nabla s\|_2^2\right).\tag{3.4}$$

Summing up (3.2), (3.3) and (3.4), and applying Gronwall's lemma yields that

$$\begin{aligned}
&(\|\nabla u\|_2^2+\|\nabla b\|_2^2)+\int_{T_0}^{t}(\|\Delta u\|_2^2+\|\Delta b\|_2^2)\,d\tau\\
&\quad\leq\left(\|\nabla u(T_0)\|_2^2+\|\nabla b(T_0)\|_2^2+\int_{T_0}^{t}(\|\nabla\theta\|_2^2+\|\nabla s\|_2^2)\,d\tau\right)\\
&\qquad\times\exp\left(C\int_{T_0}^{t}\left(1+\|\nabla_h\widetilde{u}\|_{BMO^{-1}}^2+\|\nabla_h\widetilde{b}\|_{BMO^{-1}}^2\right)d\tau\right)\\
&\quad\leq C(T_0)\exp\left(C\int_{T_0}^{t}\left(\|\nabla_h\widetilde{u}\|_{BMO^{-1}}^2+\|\nabla_h\widetilde{b}\|_{BMO^{-1}}^2\right)d\tau\right),
\end{aligned}\tag{3.5}$$

where we have used the basic energy inequality (3.1). Thanks to $\dot{B}^{-1}_{\infty,2}\hookrightarrow BMO^{-1}$ and Lemma 2, we obtain from (3.5) that

$$\begin{aligned}
&\|\nabla u\|_2^2+\|\nabla b\|_2^2+\int_{T_0}^{t}(\|\Delta u\|_2^2+\|\Delta b\|_2^2)\,d\tau\\
&\leq C(T_0)\exp\left(C\int_{T_0}^{t}\left(1+\|\nabla_h\widetilde{u}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\ln\left(e+\|\nabla_h\widetilde{u}\|_{H^1}\right)\right)d\tau\right)\\
&\quad\times\exp\left(C\int_{T_0}^{t}\left(1+\|\nabla_h\widetilde{b}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\ln\left(e+\|\nabla_h\widetilde{b}\|_{H^1}\right)\right)d\tau\right)\\
&\leq C(T_0)\exp\left(C\int_{T_0}^{T}\left(\|\nabla_h\widetilde{u}\|_{\dot{B}^{-1}_{\infty,\infty}}^2+\|\nabla_h\widetilde{b}\|_{\dot{B}^{-1}_{\infty,\infty}}^2\right)\right.\\
&\qquad\left.\times\ln\left(e+\|\nabla_h\widetilde{u}\|_{H^1}+\|\nabla_h\widetilde{b}\|_{H^1}\right)d\tau\right)
\end{aligned}$$

$$\leq C(T_0)\exp\left(C\int_{T_0}^{T}\left(\|\nabla_h\widetilde{u}\|^2_{\dot{B}^{-1}_{\infty,\infty}}+\|\nabla_h\widetilde{b}\|^2_{\dot{B}^{-1}_{\infty,\infty}}\right)\ln(e+\|u\|_{H^2}+\|b\|_{H^2})d\tau\right)$$
$$\leq C(T_0)\left(e+X(t)\right)^{C\epsilon}. \tag{3.6}$$

Next, we show the H^2 estimate. Taking Λ^2 to $(1.1)_1$ and $(1.1)_2$, and multiply the resulting equations by $\Lambda^2 u$ and $\Lambda^2 b$, respectively. Integrating over whole space and summing up the resulting identities gives that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\Lambda^2u(t)\|_2^2+\|\Lambda^2b(t)\|_2^2)&+(\|\Lambda^3u(t)\|_2^2+\|\Lambda^3u(t)\|_2^2)\\ =&-\int_{\mathbb{R}^3}\Lambda^2(u\cdot\nabla u)\cdot\Lambda^2udx+\int_{\mathbb{R}^3}\Lambda^2(b\cdot\nabla b)\cdot\Lambda^2udx\\ &+\int_{\mathbb{R}^3}\Lambda^2(\theta-s)\mathbf{k}\cdot\Lambda^2udx-\int_{\mathbb{R}^3}\Lambda^2(u\cdot\nabla b)\cdot\Lambda^2bdx\\ &+\int_{\mathbb{R}^3}\Lambda^2(b\cdot\nabla u)\cdot\Lambda^2bdx=\sum_{i=1}^{5}J_i.\end{aligned} \tag{3.7}$$

We now estimate each terms on the right hand side of (3.7) in view of commutator estimate. Taking into account $\nabla\cdot u=0$, and applying Lemma 3, Gagliardo-Nirenberg's inequality and basic energy inequlity (3.1), we get

$$\begin{aligned}J_1=&\int_{\mathbb{R}^3}\Lambda^2(u\cdot\nabla u)\cdot\Lambda^2udx=\int_{\mathbb{R}^3}\left(\Lambda^2(u\cdot\nabla u)-(u\cdot\Lambda^2\nabla u)\right)\cdot\Lambda^2udx\\ \leq&\left\|\Lambda^2(u\cdot\nabla u)-(u\cdot\Lambda^2\nabla u)\right\|_{4/3}\|\Lambda^2u\|_4\leq C\|\nabla u\|_2\|\Lambda^2u\|_4^2\\ \leq& C\|\nabla u\|_2\|u\|_2^{\frac{2}{12}}\|\Lambda^3u\|_2^{\frac{22}{12}}\leq C\|\nabla u\|_2^{12}+\frac{1}{8}\|\Lambda^3u\|_2^2.\end{aligned}$$

In the same way as in the estimate of J_1 we have for J_4

$$\begin{aligned}J_4=&\int_{\mathbb{R}^3}\Lambda^2(u\cdot\nabla b)\cdot\Lambda^2bdx=\int_{\mathbb{R}^3}\left(\Lambda^2(u\cdot\nabla b)-(u\cdot\Lambda^2\nabla b)\right)\cdot\Lambda^2bdx\\ \leq&\left\|\Lambda^2(u\cdot\nabla b)-(u\cdot\Lambda^2\nabla b)\right\|_{4/3}\|\Lambda^2b\|_4\\ \leq& C\left(\|\nabla u\|_2\|\Lambda^2b\|_4+\|\Lambda^2u\|_4\|\nabla b\|_2\right)\|\Lambda^2b\|_4\\ \leq& C(\|\nabla u\|_2+\|\nabla b\|_2)(\|\Lambda^2u\|_4^2+\|\Lambda^2b\|_4^2)\\ \leq& C(\|\nabla u\|_2+\|\nabla b\|_2)\left(\|u\|_2^{\frac{2}{12}}\|\Lambda^3u\|_2^{\frac{22}{12}}+\|b\|_2^{\frac{2}{12}}\|\Lambda^3b\|_2^{\frac{22}{12}}\right)\\ \leq& C(\|\nabla u\|_2+\|\nabla b\|_2)^{12}+\frac{1}{8}\left(\|\Lambda^3u\|_2^2+\|\Lambda^3b\|_2^2\right).\end{aligned}$$

Similarly, we have for J_2 and J_5 as follows:

$$\begin{aligned}J_2+J_5=&\int_{\mathbb{R}^3}\Lambda^2(b\cdot\nabla b)\cdot\Lambda^2udx+\int_{\mathbb{R}^3}\Lambda^2(b\cdot\nabla u)\cdot\Lambda^2bdx\\ =&\int_{\mathbb{R}^3}(\Lambda^2(b\cdot\nabla b)-(b\cdot\Lambda^2\nabla b))\cdot\Lambda^2udx+\int_{\mathbb{R}^3}(\Lambda^2(b\cdot\nabla u)-(b\cdot\Lambda^2\nabla u))\cdot\Lambda^2bdx\end{aligned}$$

$$\leq \left\|\Lambda^2(b\cdot\nabla b)-(b\cdot\Lambda^2\nabla b)\right\|_{4/3}\|\Lambda^2 u\|_4+\left\|\Lambda^2(b\cdot\nabla u)-(b\cdot\Lambda^2\nabla u)\right\|_{4/3}\|\Lambda^2 b\|_4$$
$$\leq C(\|\nabla u\|_2+\|\nabla b\|_2)(\|\Lambda^2 u\|_4^2+\|\Lambda^2 b\|_4^2)$$
$$\leq C(\|\nabla u\|_2+\|\nabla b\|_2)\left(\|u\|_2^{\frac{2}{12}}\|\Lambda^3 u\|_2^{\frac{22}{12}}+\|b\|_2^{\frac{2}{12}}\|\Lambda^3 b\|_2^{\frac{22}{12}}\right)$$
$$\leq C(\|\nabla u\|_2+\|\nabla b\|_2)^{12}+\frac{1}{8}\left(\|\Lambda^3 u\|_2^2+\|\Lambda^3 b\|_2^2\right),$$

where we have used the fact that

$$\int_{\mathbb{R}^3}(b\cdot\Lambda^2\nabla b)\cdot\Lambda^2 u dx+\int_{\mathbb{R}^3}(b\cdot\Lambda^2\nabla u)\cdot\Lambda^2 b dx=0.$$

It remains to estimate the third term J_3. By virtue of Leibniz rule, we have that

$$J_3=\int_{\mathbb{R}^3}\Lambda^2(\theta-s)\mathbf{k}\cdot\Lambda^2 u dx$$
$$=-\int_{\mathbb{R}^3}\Lambda(\theta-s)\mathbf{k}\cdot\Lambda^3 u dx\leq C(\|\nabla\theta\|_2^2+\|\nabla s\|_2^2)+\frac{1}{4}\|\Lambda^3 u\|_2^2.$$

Substituting above estimates into (3.7), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^2 u(t)\|_2^2+\|\Lambda^2 b(t)\|_2^2)+(\|\Lambda^3 u(t)\|_2^2+\|\Lambda^3 u(t)\|_2^2)\\&\qquad\leq C\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right)^6+C(\|\nabla\theta\|_2^2+\|\nabla s\|_2^2).\end{aligned}\tag{3.8}$$

Integrating (3.8) over (T_0,t) in view of basic energy inequality (3.1) and (3.6), we have

$$\begin{aligned}&\|u(t)\|_{\dot H^2}^2+\|b(t)\|_{\dot H^2}^2\\&\leq\|u(T_0)\|_{\dot H^2}^2+\|b(T_0)\|_{\dot H^2}^2+C_0+C\int_{T_0}^t\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right)^6 d\tau\\&\leq\|u(T_0)\|_{\dot H^2}^2+\|b(T_0)\|_{\dot H^2}^2+C_0+C(T_0)\int_{T_0}^t(e+X(\tau))^{6C\epsilon}d\tau.\end{aligned}\tag{3.9}$$

Taking $0<\epsilon\leq\frac{1}{6C}$, and combining (3.9) with the basic energy inequality yields that

$$X(t)\leq X(T_0)+C_0+C(T_0)\int_{T_0}^t(e+X(\tau))d\tau.$$

Applying Gronwall's inequality, one concludes that

$$X(t)\leq C(T_0)+C_0<\infty,\ \text{for all}\ t\in[T_0,T],$$

which together with (3.6) implies that

$$u,b\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)).$$

This completes the proof of the Theorem 1. $\square$

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