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On the Spectrum Structure for one Difference Eigenvalue Problem with Nonlocal Boundary Conditions

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Abstract. The difference eigenvalue problem approximating the one-dimensional differential equation with the variable weight coefficients in an integral conditions is considered. The cases without negative eigenvalue in the spectrum of difference eigenvalue problem were analyzed. Analysis of the conditions of stability of difference schemes for parabolic equations was carried out according to the theoretical results and results of the numerical experiment.

Keywords: difference eigenvalue problem, nonlocal boundary conditions, stability of difference schemes.

AMS Subject Classification: 65M06; 65M12; 65N25.

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1 Introduction and problem statement

In this paper, the difference eigenvalue problem

$$\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}+\lambda u_i=0, \quad i=1,2,\ldots,N-1, \tag{1.1}$$

$$u_0=\gamma_1[\alpha,u], \quad u_N=\gamma_2[\beta,u], \tag{1.2}$$

where $h=\frac{1}{N}$, γ_1 and γ_2 are constants,

$$[v,w]=h\Big(\frac{v_0w_0+v_Nw_N}{2}+\sum_{i=1}^{N-1}v_iw_i\Big) \tag{1.3}$$

is considered.

From theoretical point of view, this problem is a difference analog of the differential eigenvalue problem

$$\frac{d^2u}{dx^2}+\lambda u=0, \quad x\in(0,1), \tag{1.4}$$

$$u(0)=\gamma_1\int_0^1\alpha(x)u(x)dx, \quad u(1)=\gamma_2\int_0^1\beta(x)u(x)dx, \tag{1.5}$$

where $\alpha(x)$ and $\beta(x)$ are the known functions.

As far as it is known for the authors, the nonlocal conditions (1.5) for the first time were formulated in the paper [7] for one dimensional parabolic equation considering mathematical models in thermoelasticity and thermodynamics.

In [12, 17, 22] the differential eigenvalue problem with multipoint boundary conditions

$$u(0)=0, \quad u(1)=\sum_{k=1}^{m}\alpha_k u(\eta_k), \tag{1.6}$$

instead of the integral conditions (1.5) was considered.

The results of the investigation of the structure of spectrum were applied for the study of the existence and multiplicity of the nodal solution for the second-order nonlinear differential equations.

One of the main purposes of these investigations, is to determine the conditions under which the spectrum of the corresponding eigenvalue problem contains only real eigenvalues. The typical restrictions are

$$\alpha_k>0, \|\alpha\|=\left(\sum_{k=1}^{m}\alpha_k^2\right)<1.$$

See [33] for further extension to the case of integral conditions.

We note, that the eigenvalue problems (1.4)–(1.5) or (1.4)–(1.6) can be analyzed using elementary method. However, as noted in [12, 17], the corresponding eigenvalue theory is incomplete. The main reason is that the linear operators are no longer symmetric with respect to nonlocal boundary conditions (1.5)–(1.6).

We note that the difference problem (1.1)–(1.2) usually occurs when the differential equations of various types with nonlocal conditions (1.5) are solved by the finite difference method. In this sense the spectrum analysis of the problem (1.1)–(1.2) could be interpreted as one of main approaches for the investigation of stability of difference schemes and convergence of iterative methods. From the seventies and eighties of last century intensive investigation of differential equations with nonlocal conditions should be admitted. This process was stimulated by formulation of new mathematical models with nonlocal conditions in various areas of science and technology. The review of such models is provided in many papers (see, f.e. [8, 18, 30]). The investigation and methods of solution of differential equations with nonlocal conditions became one of the modern trends in the theory of differential equations and numerical analysis.

The structure of spectrum of difference eigenvalue problem with various types of nonlocal conditions was started to analyze in connection with investigation of stability of difference schemes for parabolic equations [3, 13].

Stability of the difference scheme for one-dimensional linear parabolic equation with nonlocal integral conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1,$$
$$u(0,t) = \int_0^1 \alpha(x) u(x,t) dx + \mu_1(t), \quad u(1,t) = \int_0^1 \beta(x) u(x,t) dx + \mu_2(t)$$

was investigated in many papers without the study of spectrum structure of corresponding eigenvalue problem (1.1)–(1.2) (see, f.e. [1, 6] and the references therein). The other approach oh investigation of this stability is the spectrum structure of the matrix of difference equation system [3, 15, 19, 24, 25]. One of the purposes of our investigations is to compare of these two approaches of analyzing the stability conditions. We note that for proof of stability it is necessary to determine the conditions under which the inequality $Re\lambda \geq 0$ is satisfied.

As separate mathematical task the difference eigenvalue problem with nonlocal conditions was investigated in [2, 23, 28] (see also the review article [30]). Some recent results are presented in the papers [2, 5, 10, 20, 21, 29, 31, 33].

The main purpose of the present paper is to provide new statements on the structure of spectrum of difference eigenvalue problem (1.1)–(1.2) and to apply these results for the investigation of stability of difference schemes. The theoretical results were supplemented by numerical experiment.

The structure of the paper is as following. In Section 2, we provide new theoretical results on the structure of spectrum for the problem (1.1)–(1.2) depending on the properties of variable coefficients α_i and β_i , $i = \overline{0, N}$. These theoretical statements we append in Section 3 by the results of numerical experiment. Further, in Section 4, the results of investigation of the structure of spectrum are interpreted and applied for the analysis of stability of difference schemes. Section 5 is intended to concluding remarks.

2 Difference eigenvalue problem: theory

Let the following assumptions are satisfied:

H1. $|\gamma_1\alpha_i| \le M_1 < \infty$, $|\gamma_2\beta_i| \le M_1 < \infty$, $i = \overline{0,N}$.

H2. Grid step $h \le 1/(2\gamma_k M_1)$, $k = 1, 2$.

Then we write down the eigenvalue problem (1.1)–(1.2) in the equivalent matrix form. With this aim from the conditions (1.2) we express u_0 and u_N by other unknowns

$$u_0 = \gamma_1 h \sum_{i=1}^{N-1} \tilde{\alpha}_i u_i, \quad u_N = \gamma_2 h \sum_{i=1}^{N-1} \tilde{\beta}_i u_i, \tag{2.1}$$

where

$$\tilde{\alpha}_i = \frac{1}{D}\left(\alpha_i - \frac{\gamma_2 h \alpha_i \beta_N}{2} + \frac{\gamma_2 h \alpha_N \beta_i}{2}\right), \tilde{\beta}_i = \frac{1}{D}\left(\beta_i - \frac{\gamma_1 h \beta_i \alpha_0}{2} + \frac{\gamma_1 h \alpha_i \beta_0}{2}\right),$$
$$D = \begin{vmatrix} 1 - \frac{1}{2}\gamma_1 h \alpha_0 & -\frac{1}{2}\gamma_1 h \alpha_N \\ -\frac{1}{2}\gamma_2 h \beta_0 & 1 - \frac{1}{2}\gamma_2 h \beta_N \end{vmatrix}.$$

If assumptions H1 and H2 are satisfied, then $D \ge 1 - hM_1 > 0$.

Putting expressions (2.1) for u_0 and u_N into equation (1.1), when $i = 1$ and $i = N - 1$, we get

$$Au = \lambda u,$$

where

$$A = h^{-2} \begin{pmatrix} 2 - h\tilde{\alpha}_1 & -1 - h\tilde{\alpha}_2 & -h\tilde{\alpha}_3 & \dots & \dots & -h\tilde{\alpha}_{N-2} & -h\tilde{\alpha}_{N-1} \\ -1 & 2 & -1 & \dots & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -h\tilde{\beta}_1 & -h\tilde{\beta}_2 & -h\tilde{\beta}_3 & \dots & -h\tilde{\beta}_{N-3} & -1-h\tilde{\beta}_{N-2} & 2-h\tilde{\beta}_{N-1} \end{pmatrix} \tag{2.2}$$

Now we get the following conclusion:

Corollary 1. If the assumptions H1 and H2 are fulfilled, then the difference eigenvalue problem (1.1)–(1.2) is equivalent to the eigenvalue problem for the matrix A defined by formula (2.2).

Particularly, it means that under the assumptions H1 and H2 the difference eigenvalue problem (1.1)–(1.2) has $N - 1$ eigenvalues.

Now, we will prove some statements on spectrum structure for the eigenvalue problem (1.1)–(1.2).

Theorem 1. *If (i)* $\alpha_i = 0$, $i = \overline{0,N}$, *(ii)* $[\beta, 1] < 1$, $\beta_i \ge 0$, $i = \overline{0,N}$, *then all eigenvalues of difference eigenvalue problem* (1.1)–(1.2) *are positive and different for* $0 \le \gamma_2 \le 1$.

Proof. When $\lambda > 0$ in the Equation (1.1), then $1 - \frac{1}{2}\lambda h^2 < 1$. Let us find firstly such eigenvalues $\lambda > 0$ of the problem (1.1)–(1.2), for which more strict inequality is fulfilled:

$$\left|1 - \frac{1}{2}\lambda h^2\right| < 1. \tag{2.3}$$

Taking into account inequality (2.3), we introduce new unknown s instead of λ in Equation (1.1)

$$1 - \frac{1}{2}\lambda h^2 = \cos(sh), \quad s > 0. \tag{2.4}$$

From here it follows

$$\lambda = \frac{4}{h^2}\sin^2\left(\frac{sh}{2}\right), \quad s > 0.$$

Solution of Equation (1.1), fulfilling the condition $u_0 = 0$ is

$$u_i = c\sin(sx_i), \ \ x_i = ih, \quad i = \overline{0, N},$$

where $c = \mathit{const}$. Putting this expression to the nonlocal condition (1.2) and requiring that $u_i \not\equiv 0$ we get

$$\sin(s) = \gamma_2[\beta, \sin(sx)].$$

Let us define the function

$$\Phi(s) = \sin(s) - \gamma_2[\beta, \sin(sx)].$$

If the equation $\Phi(s) = 0$ has a solution s_k, then

$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{s_k h}{2}\right) \tag{2.5}$$

is the eigenvalue of the problem (1.1)–(1.2). $\Phi(s)$ is continuous, periodic function with the period $2\pi N$. Thereby, if $\Phi(s_k) = 0$, then

$$\Phi(2\pi N - s_k) = -\Phi(s_k) = 0.$$

Further,

$$\Phi\Big(\big(k-\frac{1}{2}\big)\pi\Big) = \sin\Big(\big(k-\frac{1}{2}\big)\pi\Big) - \gamma_2\Big[\beta, \sin\Big(\big(k-\frac{1}{2}\big)\pi x\Big)\Big] > 1 - \gamma_2[\beta, 1],$$

where $k = 1, 3, 5, \ldots$. Analogously,

$$\Phi\Big(\big(k+\frac{1}{2}\big)\pi\Big) < -1 + \gamma_2[\beta, 1], \quad k = 1, 3, 5, \ldots$$

So, in every interval $I_k = \left(\left(k - \frac{1}{2}\right)\pi, \left(k + \frac{1}{2}\right)\pi\right)$, $k = 1, 2, \ldots, N-1$, equation $\Phi(s) = 0$ has at least one solution. From here it follows, that in every interval I_k, $k = \overline{1, N-1}$ the roots are different, also different are eigenvalues λ_k, calculated by formula (2.5). So, we get $N - 1$ different eigenvalues

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{N-1} < \frac{4}{h^2},$$

i.e., we obtained all eigenvalues of the $(N-1)$-th order matrix A, and at the same time eigenvalues of the problem (1.1)–(1.2). □

Remark 1. Using the analogous methodology the positive eigenvalues for the differential or difference problems to the problem (1.4)–(1.5) were obtained in [14, 22].

The following theorem is proved analogously.

Theorem 2. *If (i)* $\beta_i = 0,\ i = \overline{0,N}$, *(ii)* $[\alpha, 1] < 1$, $\alpha_i \geq 0$, $i = \overline{0,N}$, *then all eigenvalues of difference eigenvalue problem* (1.1)–(1.2) *are positive and different for* $0 \leq \gamma_1 \leq 1$.

Proof. Analogously as for Theorem 1, the solution of the problem (1.1) with the condition $u_N = 0$ is

$$u_i = c\left(\cos(sih) - \frac{\cos(s)}{\sin(s)}\sin(sih)\right), \quad i = \overline{0,N},$$

where s is defined by the equality (2.4). Putting this expression of u_i into the condition (1.2), we get

$$1 = \gamma_1[\alpha, \cos sx] - \frac{\cos(s)}{\sin(s)}\gamma_1[\alpha, \sin sx]$$

or $\sin(s) = \gamma_1[\alpha, \sin(s - sx)]$. The further proof coincides with the proof of Theorem 1. □

Lemma 1. *The negative eigenvalue of the difference eigenvalue problem* (1.1)–(1.2)*, if it exists, is expressed by the formula*

$$\lambda_k = -\frac{4}{h^2}\sinh\left(\frac{s_k h}{2}\right), \tag{2.6}$$

where $s_k > 0$ *is the root of the equation*

$$D(s) := \begin{vmatrix} 1 - \gamma_1[\alpha, \cosh(sx)] & -\gamma_1[\alpha, \sinh(sx)] \\ \cosh(s) - \gamma_2[\beta, \cosh(sx)] & \sinh(s) - \gamma_2[\beta, \sinh(sx)] \end{vmatrix} = 0. \tag{2.7}$$

Proof. When $\lambda < 0$, then $1 - \frac{\lambda h^2}{2} > 1$ and there is a possibility to introduce the new unknown $s > 0$ instead of λ according to the formula

$$1 - \frac{1}{2}\lambda h^2 = \cosh(sh).$$

From here (2.6) follows. Putting the general solution

$$u_i = c_1\cosh(sih) + c_2\sinh(sih)$$

of the Equation (1.1) to the conditions (1.2) we get two equations

$$\begin{aligned} &c_1 = c_1\gamma_1[\alpha, \cosh(sx)] + c_2\gamma_1[\alpha, \sinh(sx)], \\ &c_1\cosh(s) + c_2\sinh(s) = c_1\gamma_2[\beta, \cosh(sx)] + c_2\gamma_2[\beta, \sinh(sx)] \end{aligned}$$

in respect of c_1 and c_2. This system has nontrivial solution (c_1, c_2), if and only if the determinant of it equals to zero, i.e., the equality (2.7) is satisfied. □

Theorem 3. *The difference eigenvalue problem* (1.1)–(1.2) *has no negative eigenvalue if any of three following conditions: (i)* $\alpha_i = 0$, $\gamma_2\beta_i \le 0$, $i = \overline{0,N}$, *(ii)* $\beta_i = 0$, $\gamma_1\alpha_i \le 0$, $i = \overline{0,N}$, *(iii)* $\gamma_2\beta_i = c\gamma_1\alpha_i \le 0$, $c > 0$ *are satisfied.*

Proof. Let us take the first assumption. With this assumption we get from (2.7)

$$D(s) = \sinh(s) - \gamma_2[\beta, \sinh(sx)].$$

As $s > 0$, $\gamma_2\beta_i \le 0$, it follows from here, that $D(s) > 0$, thus equation $D(s) = 0$ has no positive roots $s > 0$.

Analogously, if the assumption (ii) is fulfilled, then we get from (2.7) that

$$\begin{aligned} D(s) =&(1 - \gamma_1[\alpha, \cosh(sx)]) \sinh(s) + \gamma_1[\alpha, \sinh(sx)] \cosh(s) \\ =& \sinh(s) - \gamma_1[\alpha_i, \sinh(sx)] > 0, \end{aligned}$$

when $s > 0$, $\gamma_1\alpha_i \le 0$. So, $D(s) = 0$ has no positive roots $s > 0$.

Now, we take assumption (iii). It follows from (2.7)

$$\begin{aligned} D(s) =& \sinh(s) - \sinh(s)\gamma_1[\alpha, \cosh(sx)] - c\gamma_1[\alpha, \sinh(sx)] \\ &+ \cosh(s)\gamma_1[\alpha, \sinh(sx)] = \sinh(s) - \gamma_1[\alpha, \sinh(s - sx)] - c\gamma_1[\alpha, \sinh(s)] > 0, \end{aligned}$$

when $s > 0$, $\gamma_1\alpha_i \le 0$, $c > 0$. Equation $D(s) = 0$ has no positive roots $s > 0$. □

In the paper [20], there are investigated the conditions of various type, under which the negative eigenvalues exist.

We admit, that considering the structure of spectrum of any eigenvalue difference problem with nonlocal conditions, much of information could be received after detailed analysis of the conditions of existence or nonexistence of the eigenvalue $\lambda = 0$. Let us carry out this for the problem (1.1)–(1.2).

Lemma 2. *[24]. The number* $\lambda = 0$ *is an eigenvalue of the difference eigenvalue problem* (1.1)–(1.2) *if and only if*

$$\gamma_1\gamma_2 \left([\alpha, x][\beta, 1] - [\beta, x][\alpha, 1]\right) + \gamma_1[\alpha, 1 - x] + \gamma_2[\beta, x] - 1 = 0. \tag{2.8}$$

Based on this lemma, we formulate few conclusions on the properties of spectrum of problem (1.1)–(1.2).

In general, formula (2.8) describes the hyperbola in the coordinate plane (γ_1, γ_2). Depending on concrete values of α_i and β_i this hyperbola might turn (degenerate) to line or two lines intersect each other by steep angle.

Two branches of hyperbola divide whole coordinate plane to three unbounded area or two areas when characteristic curve (2.8) describes the line. The part of coordinate plane to which the point of origin of coordinates belongs we mark as S_0, the rest parts of the plane are denoted as S_1 and S_2 (Figure 1).

So, in the point of origin of coordinates ($\gamma_1 = 0$, $\gamma_2 = 0$) all $N-1$ eigenvalues of the eigenvalue problem (1.1)–(1.2) are positive (the boundary conditions are of Dirichlet type). As any eigenvalue of matrix is continues function of matrix elements, the following statement is true:

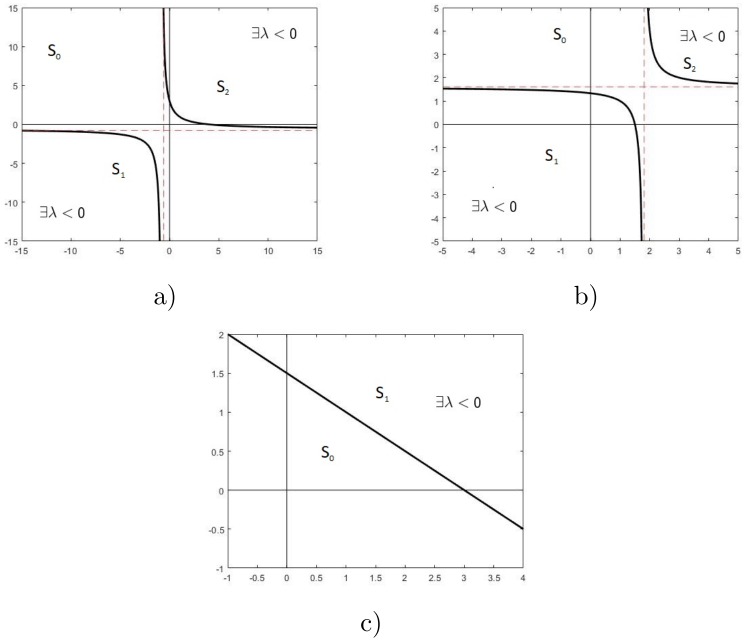


Figure 1. The schematic cases of the hyperbola (2.9) with $\alpha(x) \geq 0$, $\beta(x) \geq 0$: a) the case $A > 0$; b) the case $A < 0$; c) the case $A = 0$.

Corollary 2. Neighborhood of the point $(\gamma_1 = 0, \gamma_2 = 0)$ exists in the area S_0 such that in every point of it all the eigenvalues of problem (1.1)–(1.2) are that positive. Further, varying γ_1 and γ_2 positive eigenvalue becomes negative one when the point moves from area S_0 to the other area S_1 or S_2, crossing the hyperbola in the points of which eigenvalue $\lambda = 0$ exists.

But if in the area S_0 there are complex eigenvalues, then varying γ_1 and γ_2 positive eigenvalue could becomes negative continuously moving not through eigenvalue $\lambda = 0$, but through imaginary eigenvalue $\pm iIm\lambda$, $i = \sqrt{-1}$.

The samples of this situation are provided in [25, 26]. So, we formulate the following conclusion.

Corollary 3. If in the area S_0 or in part of it to which the point of origin of coordinates belongs, there are no complex eigenvalue of problem (1.1)–(1.2), then there are no negative eigenvalues in it, i.e., the spectrum of the problem consists only of positive numbers.

Let us consider the separate case of nonlocal conditions (1.2), $\alpha_i \geq 0$, $\beta_i \geq 0$, $i = \overline{0, N}$ often used in various types of differential equations [17, 18, 24].

When $\alpha_i \not\equiv 0$, $\beta_i \not\equiv 0$, we normalize these vectors so that would be

$$[\alpha, 1] = 1, \quad [\beta, 1] = 1.$$

After normalization we write the equation of hyperbola (2.8) as

$$A\gamma_1\gamma_2 + B\gamma_1 + C\gamma_2 - 1 = 0, \tag{2.9}$$

where $A = [\alpha, x] - [\beta, x]$, $B = 1 - [\alpha, x]$, $C = [\beta, x]$.

Now, we could show how the coefficients α_i and β_i influence on the form of hyperbola. We put out three cases:

1) $[\alpha, x] > [\beta, x]$. In this case we have: $A > 0$, $B > 0$ and $C > 0$. Therefore, the following inequalities for hyperbola asymptotes are true

$$\gamma_1 = -C/A < 0, \quad \gamma_2 = -B/A.$$

Further, when $\gamma_1 = 0$, then it follows from (2.9), that $\gamma_2 = 1/C > 0$, i.e., the point ($\gamma_1 = 0$, $\gamma_2 = 0$) is placed lower the upper branch of hyperbola (it is placed between hyperbola branches (Figure 1a)).

2) $[\alpha, x] < [\beta, x]$. In this case $A < 0$, $B > 0$, $C > 0$, the point ($\gamma_1 = 0$, $\gamma_2 = 0$) is placed lower of both of the hyperbola branches (Figure 1b).

3) $[\alpha, x] = [\beta, x]$. In this case hyperbola degenerates to the straight line:

$$B\gamma_1 + C\gamma_2 = 1,$$

$B > 0$, $C > 0$ and the point ($\gamma_1 = 0$, $\gamma_2 = 0$) is placed lower this straight line (Figure 1c).

So, we get the following statement:

Corollary 4. If $\alpha_i \geq 0$, $\beta_i \geq 0$ ($\alpha_i \not\equiv 0$, $\beta_i \not\equiv 0$), then the geometric location of eigenvalue $\lambda = 0$ of the eigenvalue problem (1.1)–(1.2) in the coordinate plane (γ_1, γ_2) is fully described by these three different types (Figures 1a, 1b, 1c). In any case the point of origin of coordinates ($\gamma_1 = 0$, $\gamma_2 = 0$) cannot be placed above the hyperbola or straight line.

3 Difference eigenvalue problem: numerical experiment

It was mentioned earlier, that the difference eigenvalue problem (1.1)–(1.2) is not investigated theoretically in complete yet. To receive new information on the structure of spectrum of this problem we performed numerical experiment. The fact that numerical experiment might complement theoretical results was demonstrated in [25, 27].

Main aims of numerical experiment are as follows:

- to widen and supplement theoretical results on the structure of spectrum depending on α_i, β_i, $i = \overline{0, N}$;

- to find some regularities characteristic to whole problem (1.1)–(1.2), not only for particular values of α_i, β_i (these regularities we formulate as corollaries).

In the numerical experiment, α_i and β_i were defined as values of continuous functions $\alpha(x)$ and $\beta(x)$ in the interval $[0, 1]$, where

$$\alpha(x), \beta(x) := \left\{1, 2x, 2(1-x), 3x^2, 3(1-x)^2, \frac{2(ax+b)}{a+2b}, \frac{\pi}{2}\sin(\pi x)\right\}. \quad (3.1)$$

Choosing the pair of functions $\alpha(x)$ and $\beta(x)$, we calculate approximately all eigenvalues of matrix A defined by formula (2.2) in the interval $\gamma_i \in [a_i, b_i]$ with some steps $\delta\gamma_1$ and $\delta\gamma_2$. The largest interval was $\gamma_i \in [-200, 200]$, the smallest one $\gamma_i \in [0, 10]$. The step of the grid was taken $h = \frac{1}{N}$, $N = 50, 100, 200, 400, 1000$. With every pair of the functions $\alpha(x)$ and $\beta(x)$ after obtaining all the eigenvalues of matrix A the structure of spectrum was represented in the coordinate plane (γ_1, γ_2).

In the graphical representation colors and significations were chosen as follows:

- A white area corresponds to the values of γ_1, γ_2 for which all eigenvalues of matrix A are real and positive.

- An area, criss-crossed with vertical and horizontal lines on a white background corresponds to the values of γ_1, γ_2 for which there exist complex eigenvalues such that $Re\lambda > 0$. The rest of eigenvalues are real and positive.

- A non criss-crossed area with a dark grey background corresponds to the values of γ_1, γ_2 for which one negative eigenvalue there exist and the rest of eigenvalues are positive. Criss-crossing of an area with sloped lines shows that there are some complex eigenvalues here.

- A non criss-crossed area on a light grey background corresponds to these values of γ_1, γ_2 for which there exist two negative eigenvalues and the rest of eigenvalues are positive. Criss-crossing of an area with sloped lines shows that there are some complex eigenvalues here.

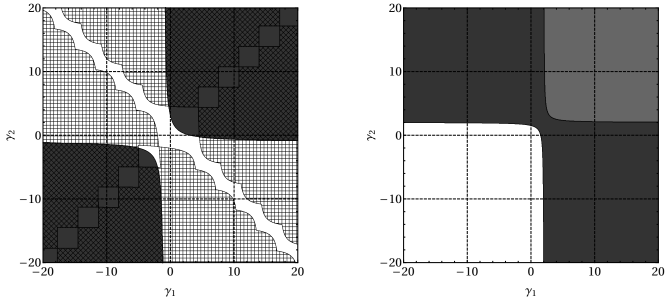
  In other words, a white background (with or without criss-crossing) sugests that the entire spectrum is such that $Re\lambda > 0$. In the areas with light grey or dark grey background the inequality $Re\lambda < 0$ is true at least for one eigenvalue.

From the results of numerical experiment, we always determine in which areas of the coordinate plane all the eigenvalues of matrix A are characterized by the property $Re\lambda > 0$. Inequality $Re\lambda > 0$ is important because it is one of conditions of the stability of difference schemes (see Section 4, formula (4.7)).

According to the results of the numerical experiment, some regularity was observed. The first elementary but quite important conclusion is following:

Corollary 5. Interchanging the functions $\alpha(x)$ and $\beta(x)$ in the eigenvalue problem (1.1)–(1.2), the structure of the spectrum may change significantly (Figure 2a, Figure 2b).

Really, when $[\alpha, x] \neq [\beta, x]$, then according to Corollary 4, the location of hyperbola in the coordinate plane (γ_1, γ_2) is different for respective pairs $(\alpha(x), \beta(x))$ and $(\beta(x), \alpha(x))$ (see Figure 1a, Figure 1b). For the pair of functions $\alpha(x) = 3x^2$, $\beta(x) = 2(1-x)$ the point $(\gamma_1 = 0, \gamma_2 = 0)$ is placed between branches of hyperbola (Figure 1a, Figure 2a). And for the pair of functions $\alpha(x) = 2(1-x)$, $\beta(x) = 3x^2$ this point is placed lower of both the branches of hyperbola (Figure 1b, Figure 2b). Namely this fact determines that in the



a) $\alpha(x) = 3x^2$, $\beta(x) = 2(1-x)$b) $\alpha(x) = 2(1-x)$, $\beta(x) = 3x^2$

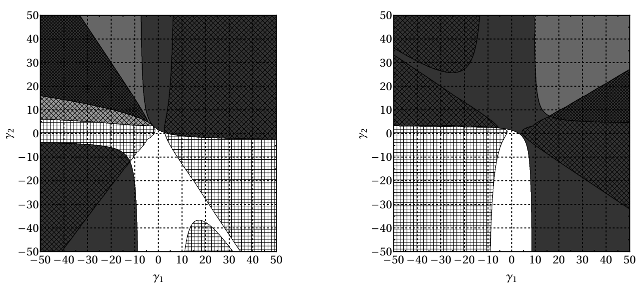
Figure 2. Effects of interchanging the functions $\alpha(x)$ and $\beta(x)$. The property $Re\lambda > 0$ is true in the points (γ_1, γ_2) of the area with white bacground (with or without criss-crossing).

cases ($\alpha(x)$, $\beta(x)$), and ($\beta(x)$, $\alpha(x)$) the areas in the coordinate plane (γ_1, γ_2) with the property $Re\lambda > 0$ are different.

In the first case (Figure 2a) the points (γ_1, γ_2) in which the property $Re\lambda > 0$ is true basically belong to the second and fourth quadrants of coordinate plane. In the second case (Figure 2b) these points with small exception belong only to the third quadrant (area S_0).

We also admit, that in the second case (Figure 2b) functions $\alpha(x)$ and $\beta(x)$ were specially selected in a way that all the eigenvalues of the problem (1.1)–(1.2) are real (there are no complex eigenvalues with all values of γ_1 and γ_2). Therefore, according to Corollary 3, all the eigenvalues in the area S_0 of the coordinate plane are positive.

The similar situation with the structure of spectrum is also in the case when complex eigenvalues exist (Figure 3a, when $\alpha(x) = 3x^2$, $\beta(x) = 2x$ and Figure 3b, when $\alpha(x) = 2x$, $\beta(x) = 3x^2$). In this case, as it was admitted in Corollary 3, all eigenvalues are positive not in the whole area S_0, but only in the part of it (Figure 3a).



a) $\alpha(x) = 3x^2,\ \beta(x) = 2x$ b) $\alpha(x) = 2x,\ \beta(x) = 3x^2$

Figure 3. Effects of interchanging the functions $\alpha(x)$ and $\beta(x)$. The property $Re\lambda > 0$ is true in the points (γ_1, γ_2) of the area with white bacground (with or without criss-crossing).

In numerical experiment, one regularity observed is directly related with the property of hyperbola (2.9) that the point of origin of the coordinate plane cannot be placed above the upper branch of hyperbola (Corollary 4). It follows from Corollary 4 that in all the points (γ_1, γ_2) placed above the hyperbola the negative eigenvalue of the problem (1.1)– (1.2) exists.

So, the following statement is true.

When (γ_1, γ_2) belongs to the first quadrant of the coordinate plane, then the property of the spectrum $Re\lambda > 0$ is characteristic only if the point (γ_1, γ_2) is close enough to the origin point of coordinates.

4 Investigation of the stability of difference schemes

We will use the results of Sections 2 and 3 for the investigation of stability of difference schemes obtained solving the parabolic equation with nonlocal conditions type (1.5) by finite difference method.

Stability of difference schemes with nonlocal boundary conditions were investigated in many papers. One of the first articles where the conditions for the stability of difference schemes for one-dimension linear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{4.1}$$

with integral conditions

$$u(0,t) = \int_0^1 \alpha(x)u(x,t)dx + \mu_1(t), \quad u(1,t) = \int_0^1 \beta(x)u(x,t)dx + \mu_2(t) \tag{4.2}$$

were investigated, is the paper [9]. Here, the stability and convergence of the forward and backward Euler methods were proved under asumptions

$$\int_0^1 |\alpha(x)|dx \le \rho < 1, \quad \int_0^1 |\beta(x)|dx \le \rho < 1. \tag{4.3}$$

The stability of Crank-Nicolson method was proved under additional asumption

$$\left(\int_0^1 \alpha^2(x)dx\right)^{\frac{1}{2}} + \left(\int_0^1 \beta^2(x)dx\right)^{\frac{1}{2}} < \frac{\sqrt{3}}{2}. \tag{4.4}$$

In [11], the stability and convergence for parabolic equation was proved then the asumptions

$$\int_0^1 \alpha^2(x)dx < 1, \quad \int_0^1 \beta^2(x)dx < 1$$

were fulfilled. In the paper [16], proving the stability of difference schemes the presumption (4.4) was replaced by another one

$$\int_0^1 \alpha^2(x)dx + \int_0^1 \beta^2(x)dx < 2, \tag{4.5}$$

which is much weaker than (4.4).

The sufficient conditions of the stability of difference scheme analogous as conditions (4.3)–(4.5) or differing from them insignificantly could be found in many other papers where the difference methods for parabolic equation with nonlocal conditions are considered [4, 6].

Beside this approach of the investigation of stability, another direction of investigation of stability of difference schemes with nonlocal conditions also was developed basing on the structure of spectrum of the matrix of difference equation system [3, 15, 19, 24, 25].

According to this approach, as an example we approximate the problem (4.1), (4.2) by using the Crank-Nicolson method and get the following difference scheme for the points on n-th layer

$$\frac{u_i^n - u_i^{n-1}}{\tau} = \frac{u_{i-1}^{n-\frac{1}{2}} - 2u_i^{n-\frac{1}{2}} + u_{i+1}^{n-\frac{1}{2}}}{h^2} + f_i^{n-\frac{1}{2}}, \quad i = 1, \dots, N-1,$$
$$u_0^n = (\alpha, u^n) + \mu_1^n, \quad u_N^n = (\beta, u^n) + \mu_2^n,$$

where $u_i^{n-\frac{1}{2}} = \frac{1}{2}\big(u_i^n + u_i^{n-1}\big)$. By analogy with eigenvalue problem (1.1)-(1.3) we express u_0^n and u_N^n for these nonlocal conditions. Putting the expressions u_0^n and u_N^n into difference equations, when $i = 1$ and $i = N-1$, we can write this difference scheme in matrix form

$$u^n = Su^{n-1} + \tau f^{n-\frac{1}{2}}. \tag{4.6}$$

where $S = (I + \frac{\tau}{2}A)^{-1}(I - \frac{\tau}{2}A)$, matrix A is determined in (2.2) (for details see [24]).

Next we formulate the theoretical statement which defines the essence of this methodology. Firstly, if $Re\lambda_k(A) > 0$, $k = \overline{1, N-1}$, where matrix A is defined by (2.2), then,

$$\rho(S) = \max_{1 \le k \le N-1} |\lambda_k(S)| < 1.$$

Thus the condition

$$Re\lambda_k(A) > 0, \quad k = \overline{1, N-1} \tag{4.7}$$

is the sufficient condition of stability of the scheme (4.6). The meaning and significance of this condition in the theory of stability were deeply analyzed earlier in [32].

Now, we can compare the conditions of stability (4.3)–(4.5) and investigate how much they differ from the stability condition (4.7). With this aim we rewrite the sufficient conditions of stability (4.3)–(4.5) in a different form

$$|\gamma_1| \, \|\alpha\|_{L_1} < 1, \quad |\gamma_2| \, \|\beta\|_{L_1} < 1, \tag{4.8}$$

$$|\gamma_1| \, \|\alpha\|_{L_2} + |\gamma_2| \, \|\beta\|_{L_2} \le \sqrt{3}/2, \;\; |\gamma_1| \, \|\alpha\|_{L_2} < 1, \;\; |\gamma_2| \, \|\beta\|_{L_2} < 1,$$
$$\gamma_1^2 \, \|\alpha\|_{L_2}^2 + \gamma_2^2 \, \|\beta\|_{L_2}^2 < 2, \tag{4.9}$$

where

$$\|\alpha\|_{L_1} = \int_0^1 |\alpha(x)| dx, \quad \|\alpha\|_{L_2} = \left(\int_0^1 \alpha^2(x) dx \right)^{\frac{1}{2}}.$$

Now, we take a concrete example.

Example 1. $\alpha(x) = 2(1-x)$, $\beta(x) = 3x^2$. The following assumptions of stability are obtained with these functions:

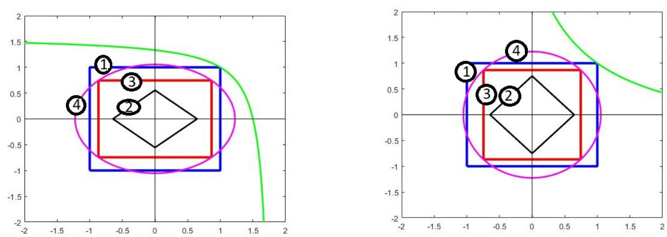
$$|\gamma_1| < 1, \quad |\gamma_2| < 1, \tag{4.10}$$

$$\frac{2}{\sqrt{3}}|\gamma_1| + \frac{3}{\sqrt{5}}|\gamma_2| < \frac{\sqrt{3}}{2}, \tag{4.11}$$

$$|\gamma_1| < \sqrt{3}/2, \quad |\gamma_2| < \sqrt{5}/3, \tag{4.12}$$

$$\frac{4}{3}\gamma_1^2 + \frac{9}{5}\gamma_2^2 < 2. \tag{4.13}$$

By these inequalities the areas of stability in the plane (γ_1, γ_2) are defined (Figure 4a). As it was mentioned before, using the coordinate plane it is convenient to interpret the results of stability. Let us take the condition of stability (4.4) [9]. The interpretation for this condition is as following. The Crank-Nicolson scheme for the Equation (4.1) with nonlocal conditions (4.2), when $\alpha(x) = 2(1-x)$, $\beta(x) = 3x^2$, will be stable, if γ_1 and γ_2 satisfy inequality (4.13), or in other words, if the point (γ_1, γ_2) belongs to the area of stability in Figure 4a, the contour of which is a curve (4.13).



a) $\alpha(x) = 2(1-x)$, $\beta(x) = 3x^2$ b) $\alpha(x) = 3x^2$, $\beta(x) = 2(1-x)$

Figure 4. The areas of the stability: 1 (blue) for the condition (4.10); 2 (black) for the condition (4.11); 3 (red) for the condition (4.12); 4 (purple) for the condition (4.13).

From Figure 4a could be seen that every out of four areas of stability, even if they are different, have many common properties. Firstly, all of them are placed in comparable small neighborhood of the point $(\gamma_1 = 0,\ \gamma_2 = 0)$, i.e., inside the circle of the radius $r = \sqrt{1.5}$. Meanwhile the real area of stability obtained during the numerical experiment (condition of stability $Re\lambda_k(A) > 0$) is incomparably larger (Figure 2b).

Secondly, the areas of stability defined by formulas (4.8)–(4.9) change little, if $\alpha(x)$ and $\beta(x)$ change each other (Figure 4b).The real area of stability obtained in the numerical experiment differs significantly.

Thirdly, the areas of stability (4.8)–(4.9) remain the same if instead of the functions $(\alpha(x),\ \beta(x))$ would be taken $(-\alpha(x),\ \beta(x))$, $(\alpha(x),\ -\beta(x))$ or $(-\alpha(x),\ -\beta(x))$. But the area of stability obtained in the numerical experiment also differs significantly.

We get analogous situation with the same conclusions if instead the functions $\alpha(x) = 2(1-x)$ and $\beta(x) = 3x^2$ we would take any functions from the set (3.1). So, from the numerical experiment it follows the conclusion.

Corollary 6. The sufficient conditions of stability of difference schemes (4.3)–(4.5) are effective enough and possibly close to the necessary conditions only for the class of functions $\alpha(x) \geq 0$, $\beta(x) \geq 0$. But if the functions $\alpha(x)$ and $\beta(x)$ are of opposite signs or each of them may change the sign, conditions (4.3)–(4.5) are ineffective and may differ much from the necessary conditions.

Any of conditions (4.3)–(4.5) may be rephrased in a following way. For difference scheme to be stable it is sufficient that norms of the $\alpha(x)$ and $\beta(x)$ in the space L_1 and L_2 would be bounded by comparably small constant (roughly not exceeding two). Even more expressive is a free interpretation of this statement. According to presumptions (4.3)–(4.5) difference scheme is stable if nonlocal conditions (4.2) differ by a little from the boundary conditions of Dirichlet type ($\alpha(x) = \beta(x) = 0$).

Numerical experiment shows that in most cases the sufficient conditions of stability (4.3)–(4.5) ineffectively enought define the real area of stability obtained on the basis of the structure of spectrum for difference problem (condition $Re\lambda(A) > 0$).

So, the values of the norms of $\alpha(x)$ and $\beta(x)$ (or shortly, magnitude of $\alpha(x)$, $\beta(x)$) often are not characteristic indication of the stability of difference scheme. More important indication of stability is structure of spectrum of difference problem. Also, it is important to emphasize that it is simpler to examine any of conditions (4.3)–(4.5) than to investigate structure of spectrum.

Example 2. Let us form and solve the differential problem in which the conditions of stability of difference scheme are not related with the magnitude of the values of $\alpha(x)$ and $\beta(x)$. Let us choose functions $\alpha(x)$ and $\beta(x)$ from the following class of functions:

$$\alpha(x) \geq 0, \quad [\alpha, 1] = 1, \quad [\beta, 1] = 1.$$

It means that some values of β_i may be negative but certain predominance of positive values is presented. Namely, let us take the following expressions of the functions $\alpha(x)$ and $\beta(x)$:

$$\alpha(x) = (10x + 4)/9, \quad \beta(x) = 10x - 4.$$

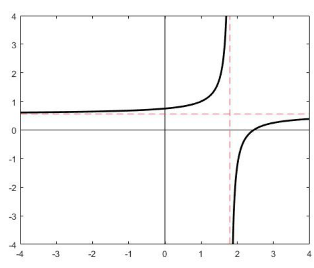


Figure 5. The location of the hyperbola (2.8) in the plane (γ_1, γ_2) in the case $\alpha(x) = (10x + 4)/9$, $\beta(x) = 10x - 4$.

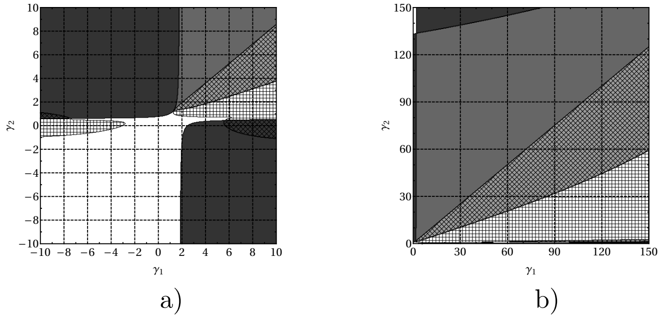


Figure 6. Spectral properties of the matrix A in the case $\alpha(x) = (10x + 4)/9$, $\beta(x) = 10x - 4$ with different intervals for γ_1, γ_2.

With these functions, the form of hyperbola (2.9) in the points of which there exists the eigenvalue $\lambda = 0$ of the problem (1.1)–(1.2) in the coordinate plane (γ_1, γ_2), is different as it was under the presence of the conditions $\alpha(x) \geq 0$, $\beta(x) \geq 0$ (Figure 5). Consequently, in the points of the first quadrant (the area S_0) of the plane (γ_1, γ_2) we could expect different properties of the spectrum. Indeed, the numerical experiment showed different structure of the spectrum than it was in the case $\alpha(x) \geq 0$, $\beta(x) \geq 0$ (see Figure 6).

We will not deal with all the properties of the structure of spectrum, but will indicate most important of them. The area of stability in the points of the first quadrant of coordinate plane (γ_1, γ_2) (criss-crossed or non criss-crossed area on a white background) resembles the centric corner or sector of circle (with some accuracy, of course). It means that it is possible to pick two neighboring points (with as big as desired values of $\gamma_1 > 0$, $\gamma_2 > 0$) such that in one of them all eigenvalues will possess a property $Re\lambda_k(A) > 0$ and in another point at least for one eigenvalue would be correct the property $Re\lambda_k(A) < 0$.

In Table 1, there are presented results received by Crank-Nicolson method for differential equation (4.1) with conditions (4.2),

$$u(x, 0) = \varphi(x),$$

where $\alpha(x) = (10x + 4)/9$, $\beta(x) = 10x - 4$.

Functions $f(x, t)$ and $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ are taken in a way that function

$$u(x, t) = e^{-t} \sin\left(\pi x/2\right)$$

would be the solution of the differential problem.

Our aim was to demonstrate that every time increasing γ_1 or γ_2 or both values we switch from the area of stability or vice versa. So, the solution of difference problem showed that the magnitude of the values $\gamma_1\alpha(x)$ and $\gamma_2\beta(x)$ alone has no influence on the stability of difference scheme. Stability depends on other properties (the sign of $Re\lambda(A)$).

Table 1. The errors of solution $\varepsilon = \max_{(i,n)} |u_i^n - u(x_i, t^n)|$ for different values h, τ, γ_1, γ_2; u_i^n - solution of the difference problem, $u(x_i, t^n)$ - solution of the difference problem.

$T = 5$ $(\gamma_1; \gamma_2)$	$h = \frac{1}{1000}$, $\tau = \frac{1}{100}$	$h = \frac{1}{4000}$, $\tau = \frac{1}{400}$	$h = \frac{1}{16000}$, $\tau = \frac{1}{1600}$
$(3; 1)$	$4.6837 \cdot 10^{-4}$	$2.9401 \cdot 10^{-5}$	$2.0155 \cdot 10^{-6}$
$(8; 3)$	$4.0537 \cdot 10^{-4}$	$2.5400 \cdot 10^{-5}$	$1.6352 \cdot 10^{-6}$
$(30; 10)$	$3.9454 \cdot 10^{-4}$	$2.4714 \cdot 10^{-5}$	$1.6141 \cdot 10^{-6}$
$(90; 30)$	$3.9099 \cdot 10^{-4}$	$2.4644 \cdot 10^{-5}$	$1.6350 \cdot 10^{-6}$
$(150; 50)$	$3.8994 \cdot 10^{-4}$	$2.4466 \cdot 10^{-5}$	$1.9582 \cdot 10^{-6}$
$(8; 4)$	$3.5937 \cdot 10^{45}$	$7.0303 \cdot 10^{115}$	$7.4685 \cdot 10^{125}$
$(30; 11)$	$4.3708 \cdot 10^{-4}$	$1.2672 \cdot 10^{13}$	$1.7646 \cdot 10^{156}$
$(90; 31)$	$3.9109 \cdot 10^{-4}$	$2.6398 \cdot 10^{-5}$	$5.4861 \cdot 10^{11}$

This effect was attained choosing $\alpha(x)$ and $\beta(x)$ in the way that at least one of these functions shift the sign of it in the interval of integration $[0, 1]$. For this reason hyperbola (2.8) in the coordinate plane (γ_1, γ_2) occupies the area (Figure 6) different from referred in Figure 1. Now the points $(\gamma_1 > 0, \gamma_2 > 0)$ of the first quadrant with some big values of γ_1, γ_2 are placed in the same area S_0 of coordinate plane as also the point $(\gamma_1 = 0,\ \gamma_2 = 0)$. We admit that the situation when $\alpha(x)$ and $\beta(x)$ shift the sign in the interval of integration may occur also in the problems of thermoelasticity [7].

5 Concluding remarks

The structure of spectrum of the problem considered in this paper plays important role in the investigation of stability of difference schemes with nonlocal conditions. To highlight this role was the main purpose of this paper.

During last three decades many different numerical methods were used to solve the parabolic equations with nonlocal conditions of various types. And many sufficient conditions of stability of difference schemes were investigated for solution of this problem. The results of the investigation of stability are described in some articles. Short overview of these investigations up to the year 2013 is provided in [1]. In the present paper we continue investigation of conditions of stability. Particularly, we emphasized the influence of the coefficients $\alpha(x)$ and $\beta(x)$ on the stability of difference schemes. For this aim we also used the numerical experiment.

More than ten years ago the author of the paper [29] wrote on the eigenvalue problem (1.4), (1.6): "Although it is important in many nonlinear problems, the corresponding eigenvalue theory for linear problem is incomplete". Today we may say analous words about the eigenvalue problem (1.1)–(1.3) and its significance in stability of difference schemes in the case of nonlocal boundary conditions.

In any way, the structure of spectrum of difference or differential eigenvalue problems (1.1)–(1.2) and (1.4)–(1.5) is an important problem of numerical analysis and differential equations and worth of further investigation.

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References

[1] B. Bialecki, G. Fairweather and J.C. Lopez-Marcos. The Crank-Nicolson Hermite cubic orthogonal spline collocation method for the heat equation with nonlocal boundary conditions. *Advan. Appl. Math. Mech.*, **54**(2013):442–460, 2013.

[2] K. Bingelė, A. Bankauskienė and A. Štikonas. Spectrum curves for a discrete Sturm-Lioville problem with one integral boundary condition. *Nonlin. Anal. Model. Control*, **24**:755–774, 2019. https://doi.org/10.15388/NA.2019.5.5.

[3] B. Cahlon, D.M. Kulkarni and P. Shi. Stepwise stability for the heat equation with a nonlocal constrain. *SIAM J. Numer. Anal.*, **32**(2):571–593, 1995. https://doi.org/10.1137/0732025.

[4] R. Čiegis, A. Štikonas, O. Štikonienė and O. Subač. A monotonic finite–difference scheme for a parabolic problem with nonlocal condition. *Differ. Equ.*, **38**(7):1027–1037, 2002. https://doi.org/10.1023/A:1021167932414.

[5] R. Čiupaila, K. Pupalaigė and M. Sapagovas. On the numerical solution for nonlinear elliptic equation with variable weight coefficients in an integral condition. *Nonlinear. Anal. Model. Contr.*, **26**(4):738–758, 2021. https://doi.org/10.15388/namc.2021.26.23929.

[6] M.R. Cui. Convergence analysis of compact difference schemes for diffusion equation with nonlocal boundary conditions. *Appl. Math. Comput.*, **260**(2015):227–241, 2015. https://doi.org/10.1016/j.amc.2015.03.039.

[7] W.A. Day. Extensions of a property of solutions of the heat equation subject to linear thermoelasticity and other theories. *Quart. Appl. Math.*, **40**:319–330, 1982. https://doi.org/10.1090/qam/678203.

[8] M. Dehghan. Efficient techniques for the second-order parabolic equation subject to nonlocal specifications. *Appl. Numer. Math.*, **52**:39–62, 2005. https://doi.org/10.1016/j.apnum.2004.02.002.

[9] G. Ekolin. Finite–difference methods for a nonlocal boundary–value problem for heat equation. *BIT*, **31**:245–261, 1991. https://doi.org/10.1007/BF01931285.

[10] N. El-Mowafy, S.M. Hedal and M.S. El-Hzab. Study the influence of nonlocal boundary condition on the difference eigenvalue problem for differential equation. *J. Informat. and Mathem. Scienc.*, **12**(3):209–222, 2020.

[11] G. Fairweather and J.C. Lopez-Marcos. Galerkin methods for a semilinear parabolic problem with nonlocal conditions. *Adv. Comp. Math.*, **6**:243–262, 1996. https://doi.org/10.1007/BF02127706.

[12] J. Gao, D. Sun and M. Zhang. Structure of eigenvalues of multi-point boundary value problems. *Advan. Difference Equat.*, **381932**(2010):1–18, 2010.

[13] N.I. Ionkin. Solution of one boundary value problem of the theory of heat condaction with a nonclasical boundary condition. *Differ. Equ.*, **13**:204–211, 1977.

[14] J. Jachimavičienė, Ž. Jesevičiūtė and M. Sapagovas. The stability of finite–difference schemes for a pseudoparabolic equations with nonlocal conditions. *Numer. Funct. Anal. Optim.*, **30**(9):988–1001, 2009. https://doi.org/10.1080/01630560903405412.

[15] T. Leonavičienė, A. Bugajev, G. Jankevičiūtė and R. Čiegis. On stability analysis of finite difference schemes for generalized Kuramoto–Tsuzuki equation with nonlocal boundary conditions. *Math. Model. Anal.*, **21**(5):630–643, 2016. https://doi.org/10.3846/13926292.2016.1198836.

[16] Y. Liu. Numerical solution of the heat equation with nonlocal boundary conditions. *J. Comput. Appl. Math*, **110**:115–127, 1999. https://doi.org/10.1016/S0377-0427(99)00200-9.

[17] R. Ma and D. O'Regan. Nodal solution for second-order m-point boundary value problems with nonlinearities across several eigenvalues. *Nonlin. Anal. Theory, Math & Appl.*, **64**(2006):1562–1577, 2006. https://doi.org/10.1016/j.na.2005.07.007.

[18] J. Martin-Vaquero and J. Vigo-Aguiar. On the numerical solution of the heat conduction equations subject to nonlocal conditions. *Appl. Numer. Math.*, **59**:2507–2514, 2009. https://doi.org/10.1016/j.apnum.2009.05.007.

[19] J. Novickij and A. Štikonas. On the stability of a weighted finite difference scheme for wave equation with nonlocal boundary conditions. *Nonlinear Anal. Model. Control*, **19**(3):460–475, 2014. https://doi.org/10.15388/NA.2014.3.10.

[20] S. Pečiulytė, O. Štikonienė and A. Štikonas. Investigation of negative critical points of the characteristic function for problem with nonlocal boundary conditions. *Nonlin. Anal. Model. Contr.*, **13**(4):467–490, 2008. https://doi.org/10.15388/NA.2008.13.4.14552.

[21] K. Pupalaigė, M. Sapagovas and R. Čiupaila. Nonlinear elliptic equation with nonlocal integral boundary condition depending on two parameters. *Math. Model. Anal.*, **27**(4):610–628, 2022. https://doi.org/10.3846/mma.2022.16209.

[22] B.P. Rynne. Spectral properties and nodal solutions for second-order boundary value problems. *Nonlin. Anal. Theory, Meth. & Applic*, **67**(12):3318–3327, 2007. https://doi.org/10.1016/j.na.2006.10.014.

[23] M. Sapagovas. The eigenvalues of some problems with a nonlocal condition. *Differ. Equ.*, **38**:1020–1026, 2002. https://doi.org/10.1023/A:1021115915575.

[24] M. Sapagovas. On the stability of a finite-difference scheme for nonlocal parabolic boundary–value problems. *Lith. Math. J.*, **48**(3):339–356, 2008. https://doi.org/10.1007/s10986-008-9017-5.

[25] M. Sapagovas, T. Meškauskas and F. Ivanauskas. Numerical spectral analysis of a difference operator with non-local boundary conditions. *Appl. Math. Comput.*, **218**(14):7515–7527, 2012. https://doi.org/10.1016/j.amc.2012.01.017.

[26] M. Sapagovas, T. Meškauskas and F. Ivanauskas. Influence of complex coefficients on the stability of difference scheme for parabolic equations with nonlocal conditions. *Appl. Math. Comp.*, **332**:228–240, 2018. https://doi.org/10.1016/j.amc.2018.03.072.

[27] M. Sapagovas, R. Čiupaila, Ž. Jokšienė and T. Meškauskas. Computational experiment for stability analysis of difference schemes with nonlocal conditions. *Informatica*, **24**(2):275–290, 2013. https://doi.org/10.15388/Informatica.2013.396.

[28] M.P. Sapagovas and A.D. Štikonas. On the stucture of the spectrum of a differential operator with a nonlocal condition. *Differ. Equ.*, **41**(7):1010–1018, 2005. https://doi.org/10.1007/s10625-005-0242-y.

[29] A. Štikonas. Investigation of characteristic curve for Sturm-Liouville problem with nonlocal boundary conditions on torus. *Math. model. Anal.*, **16**(1):1–22, 2011. https://doi.org/10.3846/13926292.2011.552260.

[30] A. Štikonas. A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions. *Nonlinear. Anal. Model. Control*, **19**(3):301–334, 2014. https://doi.org/10.15388/NA.2014.3.1.

[31] A. Štikonas and E. Şen. Asymptotic analysis of Sturm-Liouville problem with nonlocal integral type boundary condition. *Nonlin. Anal. Model. Contr.*, **26**(15):969–991, 2021. https://doi.org/10.15388/namc.2021.26.24299.

[32] R.S. Varga. Matrix Iterative Analysis. *Prentice Hall, New Jersy*, 1962.

[33] Z.C. Zhou and F.F. Liao. Structure and asymptotic expansion of eigenvalues of an integral type nonlocal problem. *Electr. J. Different. Equat*, **2016**(283):1–12, 2016.