Mathematical Modelling and Analysis
Volume 28, Issue 2, 194–217, 2023
https://doi.org/10.3846/mma.2023.15999

On Discrete-Time Models of Network Worm Propagation Generated by Quadratic Operators

Fatima Adilova[a], **Uygun Jamilov**[a,b,c] **and Andrejs Reinfelds**[d,e]

[a] *V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences*
University str. 9, 100174 Tashkent, Uzbekistan
[b] *Akfa University*
National Park str. 264, Qibray district, 111221 Tashkent region, Uzbekistan
[c] *National University of Uzbekistan*
University str. 4, 100174 Tashkent, Uzbekistan
[d] *University of Latvia*
Jelgavas str. 3, 1001 Riga, Latvia
[e] *Institute of Mathematics and Computer Sciences*
Raine Bulvaris 29, 1459 Riga, Latvia
E-mail(*corresp.*): uygun.jamilov@mathinst.uz, jamilovu@yandex.ru
E-mail: fatadilova@gmail.com
E-mail: andrejs.reinfelds@lu.lv, reinf@latnet.lv

Received November 22, 2021; revised December 26, 2022; accepted January 2, 2023

Abstract. In this paper we consider the discrete-time dynamical systems generated by network worm propagation models based on the theory of quadratic stochastic operators(QSO). This approach simultaneously solves two important problems: exploring of the QSO trajectory‘s behavior, we described the set of limit points, thereby completely solved the main problem of dynamical systems (i), we showed a new application of the theory QSOs in worm propagation modelling (ii). We demonstrated that proposed discrete-time biologically-inspired model represents also realistic picture of the worm propagation process and such analytical models can be used in decision of some problems of computer networks.

Keywords: network worms, propagation dynamics, modeling, quadratic stochastic operator, regular operator.

AMS Subject Classification: 37N25; 92D10; 68M07.

1 Introduction

Attacks of network worms are actual problems of network security. They always cause direct financial damage and also serves as the basis, for example, stealing of confidential and private personal information, etc.

As a network worm can be understand as a program capable of independently finding new nodes for infection and using a communication network for its propagation. Existing defenses do not always efficiently cope with the epidemics of network worms; therefore, it becomes urgent to create a new generation of detection and protection systems that can prevent or contain the epidemic in its early stages. To solve this problem, it is necessary to be able to model the epidemics of network worms in order to deep study this phenomenon in details and the model should adequately describe the process of spreading network worms. A powerful approach for analyzing the spread of computer worms across the Internet is analytical modeling. An advantage of analytical modeling is traditionally considered to be a solution in general, as well as a high speed of modeling specific scenarios for various initial conditions.

The mathematical instruments for studying the dynamics of biological epidemics have been developed a long time ago. Various models borrowed from epidemiology have already been used to model the dynamics of the spread of Internet worms. The biological approach to modeling a viral problem admittedly began with the work [16], but this topic becomes relevant after outbreaks of Code Red and Nimda in 2001. We refer reader to [5] and corresponding references therein for information about worms Code Red and Nimda.

In [20] the SIS models are reviewed and the N-intertwined virus spread model of the SIS-type is introduced as a promising and analytically tractable model of which the steady-state behavior is fairly completely determined. Also it is compared to the exact SIS Markov model, the N-intertwined model makes only one approximation of a mean-field kind that results in upper bounding the exact model for finite network size N and improves in accuracy with N.

In [7] an e-epidemic SIR compartmental model is considered, i.e. the population of hosts is divided into three classes: susceptible computers, infected computers and recovered computers, and the evolution between these states is ruled according to specific local transition functions involving Boolean expressions.

In [5, 6] the SEIR-SEIQR-SAIR-PSIDR models were studied and their results applied Code Red Worm propagation and the PSIDR model showed results closest to real data.

In all of the above-mentioned references the authors investigated continuous time dynamical systems generated by SIS-SIR-SEIR-SEIQR-SAIR-PSIDR models. However, it seems natural to consider a discretization of these continuous time systems and explore the discrete time dynamical systems resulting from it. Especially it is known that some discrete and the corresponding continuous time dynamical systems behavior substantially differently. Indeed, the authors showed in [1] that the discrete time dynamical system is much richer than the continuous time dynamical system generated by same operator. Also in [2] the behavior in the discrete-time SIS and SIR models with some type of positive feedback to the susceptible class differs from their continuous analogues is proven.

The main aim of the research is to study the discrete-time dynamical systems of the above mentioned SIS-SIR-PSIRD models using the theory of quadratic stochastic operators. Therefore the discrete time dynamical system of network worm propagation, presented in QSO form of SIS-SIR-PSIRD models in the present paper are interesting, because they distinguish from previous discrete models in some basic features. Firstly, we considered the models in the two- and three -dimensional simplexes; secondly, two models (PSIDR, SIR) considered as the case when the number of nodes is variable; thirdly, exploring of the trajectory's behavior, we described the set of limit points, thereby completely solved the main problem of modelling. And finally, we showed a new application of the theory QSOs. In our investigation, some parameters can be negative which are impossible in real situations, but these cases are also interesting from a purely mathematical point of view. Such dynamical systems are important, but there are relatively few dynamical phenomena that are currently understood (see e.g. [25]).

The paper organized as follow. In the next Section 2, we recall the definitions SIS-SIR- PSIDR models and quadratic stochastic operators. In Section 3, we reduce SIS-SIR models to a quadratic stochastic operator mapping the simplexes S^1 and respectively S^2 to itself. It is described the set of limit points the trajectories depending on the parameters of model and on the initial points. Section 4 is devoted dynamical system generated by discrete version of PSIDR model. It is shown that the trajectories of discrete time SIS-SIR- PSIDR models (operators) are convergent.

2 Preliminaries and known results

In this section, following [18], we describe the models of network worm propagation: *Susceptible-Infected-Susceptible* (SIS) model: Let an arbitrary host of a network consisting of a constant number of N hosts be in two states - vulnerable (S) and infected (I), i.e. $S + I = N$. Suppose that when a host is infected, only one copy of the worm can exist on it. This copy randomly selects a potential victim in the accessible address space with an average constant speed β hosts per second. On average, $1/\beta$ seconds is spent on searching and infecting a single host. To describe the dynamics of the share of infected hosts, we introduce the variables $i = I/N$ and $s = S/N$. The equations of the dynamics of such system has following form:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\beta is, \qquad \frac{\mathrm{d}i}{\mathrm{d}t} = \beta is. \tag{2.1}$$

Susceptible-Infected-Removed (SIR) model: a) The SIR model can be used to estimate factors that attenuate network outbreaks. In it, hosts exist in three states: vulnerable (S), infected (I) and immune (R). Thus, $S + I + R = N$. Suppose that the nodes are invulnerable only after the treatment of infection.

Introducing a constant average speed of "immunization" per unit time γ and given that $i = I/N$, $r = R/N$ and $s = S/N$, we get the system of equations

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= -\beta is, \\ \frac{\mathrm{d}i}{\mathrm{d}t} &= \beta is - \gamma i, \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \gamma i. \end{aligned} \tag{2.2}$$

b) The SIR model with a variable number of nodes can be considered similarly. The dynamics of a system with a variable number of nodes is determined by the rate of growth of new vulnerable (S) nodes α:

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= -\beta is - (\alpha + \gamma)s + \alpha, \\ \frac{\mathrm{d}i}{\mathrm{d}t} &= \beta is - (\alpha + \gamma)i, \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \gamma(1 - r) - \alpha r. \end{aligned} \tag{2.3}$$

Progressive Susceptible-Infected-Detected-Removed (PSIDR) model: In the PSIDR model, it is assumed that epidemic events are divided into two periods.

The pre-response period. Initially, the worm infects one host on the network. For several days (hours), the worm spreads over the network without being noticed by most users. In terms of the PSIDR model, this phase is characterized by a positive infection rate (worm birth) β without cure attempts. Vulnerable nodes become infected with a probability of β if they are in contact with an infected node.

The response period. After a period of time, the worm is detected on some hosts. Its signatures are highlighted and entered into the anti-virus software database. Uninfected nodes become immune to this worm, and infected hosts are "cured" with a certain frequency, depending on the update rate of the anti-virus database. This period in the model under consideration is characterized by the same frequency of birth of the worm, but vulnerable nodes are cured with the frequency μ, and already infected nodes are detected with the frequency μ and cured with the frequency δ. The parameter μ essentially characterizes the speed at which the update of the anti-virus database spreads after the initial detection of the worm.

Thus, the PSIDR model assumes that the epidemic can be divided into two periods and it is assumed that the number of nodes in the network N is constant. When $0 < t < \pi$ it is must be $S(t) + I(t) = N$. We introduce the variables $s = S/N$ and $i = I/N$ then the system is described by the following equations

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\beta is, \qquad \frac{\mathrm{d}i}{\mathrm{d}t} = \beta is. \tag{2.4}$$

When $t \geq \pi$ it is must be $S(t) + I(t) + D(t) + R(t) = N$. We introduce the variables $s = S/N$, $i = I/N$, $d = D/N$ and $r = R/N$ then the system is described by the following equations:

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= -\beta is - \mu s, \\ \frac{\mathrm{d}i}{\mathrm{d}t} &= \beta is - \mu i, \\ \frac{\mathrm{d}d}{\mathrm{d}t} &= \mu i - \delta d, \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \delta d + \mu s. \end{aligned} \tag{2.5}$$

Note that in [3] the continuous time dynamical systems generated by operator (2.5) studied in details. In the below we consider the discrete time versions of SIS-SIR-PSIRD models, but generated by a quadratic stochastic operator. So here we shortly describe the essence of quadratic stochastic operator.

Quadratic stochastic operators(QSO): Let

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \ x_i \geq 0, \ \text{for any } i \ \text{and} \ \sum_{i=1}^{m} x_i = 1\}$$

be the $(m-1)$-dimensional simplex. A map V from S^{m-1} into itself is called a quadratic stochastic operator if

$$V : x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \tag{2.6}$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \ldots, m$, where

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \sum_{k=1}^{m} p_{ij,k} = 1. \tag{2.7}$$

Such operators occur in some models of mathematical biology and describe the evolution of a population consisting of several types of individuals (see e.g., [1, 4, 9, 10, 11, 12, 13, 14, 15, 17, 19, 21, 22, 23, 24, 26]). Assume that $\{\mathbf{x}^{(n)} \in S^{m-1} : n = 0, 1, 2, \ldots\}$ is the trajectory of the initial point $\mathbf{x} \in S^{m-1}$, where $\mathbf{x}^{(n+1)} = V\big(\mathbf{x}^{(n)}\big)$ for all $n = 0, 1, 2, \ldots$, with $\mathbf{x}^{(0)} = \mathbf{x}$. Let $\omega\big(\mathbf{x}^{(0)}\big)$ be the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}$, $n = 0, 1, \ldots$. The main problem is investigation of the asymptotical behaviour of the trajectories for a given QSO. In other words the main task is the description of the set $\omega\big(\mathbf{x}^{(0)}\big)$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO. This problem is an open problem even in two-dimensional case. A point $\mathbf{x} \in S^{m-1}$ is called a *fixed point* of a QSO V if $V(\mathbf{x}) = \mathbf{x}$. A QSO V is called *regular* if for any initial point $\mathbf{x} \in S^{m-1}$, the limit $\lim\limits_{n\to\infty} V^n(\mathbf{x})$ exists. Note that the limit point is a fixed point of a QSO. Thus, the fixed points of a QSO describe limit or long run behavior of the trajectories for any initial point. The limit behavior of trajectories and fixed points play an important role in many applied problems.

Definition 1. [11] A quadratic stochastic operator is called Volterra if

$$p_{ij,k} = 0, \quad \text{for any} \ \ k \notin \{i, j\}, \, i, j, k = 1, \ldots, m.$$

Fix $l \in \{1, \dots, m\}$ and assume that coefficients $p_{ij,k}$ satisfy

$$\begin{aligned} &p_{ij,k} = 0 \ \text{ if } \ k \notin \{i,j\} \ \text{ for any } \ k \in \{1,\dots,l\},\, i,j = 1,\dots,m; \\ &p_{ij,k} > 0 \ \text{ for at least one pair } \ (i,j),\, i \neq k, j \neq k, \forall k \in \{l+1,\dots,m\}. \end{aligned} \tag{2.8}$$

Definition 2. [26] For any fixed $l \in \{1, \dots, m\}$, the QSO defined by (2.6), (2.7) and (2.8) is called $l-$ Volterra QSO.

3 Discrete time SIS, SIR-models

Consider discrete time analogues of the above mentioned SIS-SIR models of worm propagation. In each model to make some relations with $p_{ij,k}$ we find conditions on parameters of models (operators) rewriting them in the form a QSO (2.6). The main task about the asymptotical behaviour of the trajectories for any QSO is the open problem still. This is a motivation of considering the discrete time analogues of these models, presenting in a QSO form.

3.1 Discrete time SIS model

The discrete version of (2.1) has the following form

$$\begin{cases} s_{n+1} = s_n - \beta s_n i_n, \\ i_{n+1} = i_n + \beta s_n i_n, \end{cases}$$

where n is a non-negative integer number.

Consider an operator $V : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $\mathbf{x} = (x_1, x_2) \to \mathbf{x}' = V(\mathbf{x}) = (x'_1, x'_2)$ defined by

$$V_1 : \begin{cases} x'_1 = x_1 - \beta x_1 x_2, \\ x'_2 = x_2 + \beta x_1 x_2, \end{cases} \tag{3.1}$$

where $x'_1 = s_{n+1}$, $x'_2 = i_{n+1}$, $x_1 = s_n$, $x_2 = i_n$ for a natural n.

The discrete time model (3.1) is related to a QSO defined on the simplex S^1. Indeed, using $x_1 + x_2 = 1$ we rewrite the operator (3.1) in the form

$$\begin{cases} x'_1 = x_1^2 + (1-\beta)x_1x_2, \\ x'_2 = x_2^2 + (1+\beta)x_1x_2, \end{cases} \tag{3.2}$$

that is the operator is a QSO with the following coefficients

$$p_{11,1} = 1, \ \ 2p_{12,1} = 1 - \beta, \ \ 2p_{12,2} = 1 + \beta, \ \ p_{22,2} = 1.$$

Therefore, the operator (3.1) maps S^1 to itself if and only if $-1 \le \beta \le 1$ and under this condition it is a Volterra QSO. The Volterra QSOs deeply studied in [9, 11, 12]. Using the results of dynamics of Volterra QSOs one can easily show that

Theorem 1. *The following statements are true for the QSO* (3.2)*:*

i) if $\beta = 0$ then V_1 is the identity map;

ii) the vertexes $\mathbf{e}_1 = (1, 0)$ *and* $\mathbf{e}_2 = (0, 1)$ *of the* S^1 *are fixed points;*

iii) if $\beta < 0$ *then* $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_1$ *for any* $\mathbf{x}^{(0)} \in S^1 \setminus \{\mathbf{e}_2\}$*;*

iv) if $\beta > 0$ *then* $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_2$ *for any* $\mathbf{x}^{(0)} \in S^1 \setminus \{\mathbf{e}_1\}$*.*

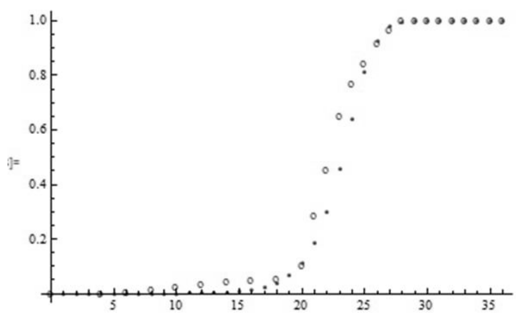

Figure 1. SIS model of Code Red propagation (black color points) and actual Code Red propagation (white color points).

Remark 1. Figure 1 shows the (3.1) SIS model's trajectory of Code Red propagation (in black color points) when $N = 350000$, $\beta = 0.75$, $I = 1$ (see [5]) $n = 36$ hours and actual Code Red propagation (in white color points).

Remark 2. In [2] the operator (3.1) is considered as a transformation which mapping the set $\mathbb{R}^2_+ = \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}$ into itself. It is proven that the discrete time dynamical systems is the same as continuous analogue.

3.2 Discrete time SIR model with the number of nodes is a constant

The discrete version of (2.2) has the following form

$$\begin{cases} s_{n+1} = s_n - \beta s_n i_n, \\ i_{n+1} = i_n - \gamma i_n + \beta s_n i_n, \\ r_{n+1} = r_n + \gamma i_n, \end{cases}$$

where n is a non-negative integer number.

Consider an operator $V : \mathbb{R}^3_+ \to \mathbb{R}^3_+$, $\mathbf{x} = (x_1, x_2, x_3) \to \mathbf{x}' = V(\mathbf{x}) = (x'_1, x'_2, x'_3)$ defined by

$$V_2 : \begin{cases} x'_1 = x_1(1 - \beta x_2), \\ x'_2 = x_2(1 - \gamma + \beta x_1), \\ x'_3 = x_3 + \gamma x_2, \end{cases} \tag{3.3}$$

where $x'_1 = s_{n+1}$, $x'_2 = i_{n+1}$, $x'_3 = r_{n+1}$, $x_1 = s_n$, $x_2 = i_n$, $x_3 = r_n$ for a natural n.

Using $x_1+x_2+x_3=1$ one can rewrite the Equation (3.3) in the form

$$\begin{cases} x_1' = x_1^2 + (1-\beta)x_1x_2 + x_1x_3, \\ x_2' = (1-\gamma)x_2^2 + (1-\gamma+\beta)x_1x_2 + (1-\gamma)x_2x_3, \\ x_3' = \gamma x_2^2 + x_3^2 + \gamma x_1x_2 + x_1x_3 + (1+\gamma)x_2x_3. \end{cases} \tag{3.4}$$

We consider QSO for the $m=3$:

$$\begin{cases} x_1'{=}p_{11,1}x_1^2{+}p_{22,1}x_2^2{+}p_{33,1}x_3^2{+}2p_{12,1}x_1x_2{+}2p_{13,1}x_1x_3{+}2p_{23,1}x_2x_3, \\ x_2'{=}p_{11,2}x_1^2{+}p_{22,2}x_2^2{+}p_{33,2}x_3^2{+}2p_{12,2}x_1x_2{+}2p_{13,2}x_1x_3{+}2p_{23,2}x_2x_3, \\ x_3'{=}p_{11,3}x_1^2{+}p_{22,3}x_2^2{+}p_{33,3}x_3^2{+}2p_{12,3}x_1x_2{+}2p_{13,3}x_1x_3{+}2p_{23,3}x_2x_3. \end{cases} \tag{3.5}$$

From the Equations (3.4) and (3.5) we have the following relations:

$$\begin{array}{rlrlrl} p_{11,1} &= 1, & 2p_{12,1} &= 1-\beta, & 2p_{13,1} &= 1, \\ p_{11,2} &= 0, & 2p_{12,2} &= 1-\gamma+\beta, & 2p_{13,2} &= 0, \\ p_{11,3} &= 0, & 2p_{12,3} &= \gamma, & 2p_{13,3} &= 1, \\ p_{22,1} &= 0, & 2p_{23,1} &= 0, & p_{33,1} &= 0, \\ p_{22,2} &= 1-\gamma, & 2p_{23,2} &= 1-\gamma, & p_{33,2} &= 0, \\ p_{22,3} &= \gamma, & 2p_{23,3} &= 1+\gamma, & p_{33,3} &= 1. \end{array} \tag{3.6}$$

Proposition 1. *The operator* (3.4) *maps* S^2 *to itself if and only if*

$$0 \le \gamma \le 1, \quad \gamma - 1 \le \beta \le 1. \tag{3.7}$$

Moreover, under condition (3.7) *the operator* (3.4) *is a 1-Volterra QSO.*

Proof. The proof consists solving of simple inequalities which are obtained from conditions (2.7) for $p_{ij,k}$ given by equalities (3.6). □

Remark 3. In [2], the operator (2.2) is considered as a transformation which mapping mapping the set $\mathbb{R}^3_+ = \{\mathbf{x} \in \mathbb{R}^3 : x_1 \ge 0,\, x_2 \ge 0,\, x_3 \ge 0\}$ into itself. It is showed that the discrete time dynamical systems is the same as continuous version.

It is evident that if $\beta = \gamma = 0$ then V_2 is the identity map. For any parameters (3.7), the set $\Gamma_{\{2\}} = \{\mathbf{x} \in S^2 : x_2 = 0\}$ consists from the fixed points of operator V_2.

I. Case $\beta = 0, \gamma \ne 0$: In this case, one has that

$$x_1' = x_1, \quad x_2' = (1-\gamma)x_2 \le x_2, \quad x_3' = x_3 + \gamma x_2 \ge x_3.$$

Consequently, it follows

$$x_1^{(n)} = x_1^{(0)}, \quad x_2^{(n)} = (1-\gamma)^n x_2^{(0)}, \quad x_3^{(n)} \ge x_3^{(n-1)}, \quad n = 1, 2, \dots$$

and existence of the following limits

$$\lim_{n\to\infty} x_1^{(n)} = x_1^{(0)}, \quad \lim_{n\to\infty} x_2^{(n)} = 0, \quad \lim_{n\to\infty} x_3^{(n)} = 1 - x_1^{(0)}.$$

Thus, $\lim_{n\to\infty}\mathbf{x}^{(n)} = \big(x_1^{(0)}, 0, 1-x_1^{(0)}\big)$ for any $\mathbf{x}^{(0)}\in S^2$ with $x_2^{(0)}>0$.

II. a) Case $\beta>0$, $\gamma=0$: In this case, one has that

$$x_1' = x_1(1-\beta x_2)\le x_1, \quad x_2' = x_2(1+\beta x_1)\ge x_2, \quad x_3' = x_3.$$

Therefore, we have

$$x_1^{(n)} \le x_1^{(n-1)}, \quad x_2^{(n)} \ge x_2^{(n-1)}, \quad x_3^{(n)} = x_3^{(0)}, \quad n=1,2,\dots.$$

Since the sequence $x_1^{(n)}$ (resp. $x_2^{(n)}$) is a monotone decreasing (resp. increasing) bounded from the below (resp. above) it follows that $\lim_{n\to\infty} x_1^{(n)} = x_1^*$ (resp. $\lim_{n\to\infty} x_2^{(n)} = x_2^*$). Evidently that $\lim_{n\to\infty} x_3^{(n)} = x_3^* = x_3^{(0)}$. So, there is the limit $\lim_{n\to\infty}\mathbf{x}^{(n)} = \mathbf{x}^*$ and it should be a fixed point. Then one easily has that $\mathbf{x}^* = \big(0, 1-x_3^{(0)}, x_3^{(0)}\big)$ for any $\mathbf{x}^{(0)}\in S^2$ with $x_2^{(0)}>0$.

b) Case $\beta<0$, $\gamma=0$: In this case, one has that

$$x_1' = x_1(1-\beta x_2)\ge x_1, \quad x_2' = x_2(1+\beta x_1)\le x_2, \quad x_3' = x_3.$$

Therefore, we have

$$x_1^{(n)} \ge x_1^{(n-1)}, \quad x_2^{(n)} \le x_2^{(n-1)}, \quad x_3^{(n)} = x_3^{(0)}, \quad n=1,2,\dots.$$

Since the sequence $x_1^{(n)}$ (resp. $x_2^{(n)}$) is a monotone increasing (resp. decreasing) bounded from the above (resp. below) it follows that $\lim_{n\to\infty} x_1^{(n)} = x_1^*$ (resp. $\lim_{n\to\infty} x_2^{(n)} = x_2^*$). It is clear that $\lim_{n\to\infty} x_3^{(n)} = x_3^* = x_3^{(0)}$. So, there is the limit $\lim_{n\to\infty}\mathbf{x}^{(n)} = \mathbf{x}^*$ and it should be a fixed point. Then one easily has that $\mathbf{x}^* = \big(1-x_3^{(0)}, 0, x_3^{(0)}\big)$ for any $\mathbf{x}^{(0)}\in S^2$ with $x_2^{(0)}>0$.

III. a) Case $\beta>0, \gamma\neq 0$: In this case, it follows

$$x_1' = x_1(1-\beta x_2)\le x_1, \quad x_2' = x_2(1-\gamma+\beta x_1), \quad x_3' = x_3+\gamma x_2 \ge x_3.$$

Therefore, one has

$$x_1^{(n)} \le x_1^{(n-1)}, \quad x_2^{(n)} = 1-x_1^{(n)}-x_2^{(n)}, \quad x_3^{(n)} \ge x_3^{(n-1)}, \quad n=1,2,\dots$$

and it follows the existence of limits $\lim_{n\to\infty} x_1^{(n)} = x_1^*$, $\lim_{n\to\infty} x_3^{(n)} = x_3^*$ and $\lim_{n\to\infty} x_2^{(n)} = 1-x_1^*-x_3^*$. So, there is the limit $\lim_{n\to\infty}\mathbf{x}^{(n)} = \mathbf{x}^*$. Since the limit point $\mathbf{x}^*$ should be a fixed point it follows that $x_2^* = 0$. Thus $\lim_{n\to\infty}\mathbf{x}^{(n)} = (x_1^*, 0, 1-x_1^*)$ for any $\mathbf{x}^{(0)}\in S^2$ with $x_2^{(0)}>0$.

b) Case $\beta<0$, $\gamma\neq 0$: In this case, it follows

$$x_1' = x_1(1-\beta x_2)\ge x_1, \quad x_2' = x_2(1-\gamma+\beta x_1)\le x_2, \quad x_3' = x_3+\gamma x_2 \ge x_3.$$

Therefore, one has

$$x_1^{(n)} \ge x_1^{(n-1)}, \quad x_2^{(n)} \le (1-\gamma)x_2^{(n-1)}, \quad x_3^{(n)} \ge x_3^{(0)}, \quad n=1,2,\dots$$

and it follows the existence of limits $\lim\limits_{n\to\infty} x_1^{(n)} = x_1^*$, $\lim\limits_{n\to\infty} x_2^{(n)} = 0$ and $\lim\limits_{n\to\infty} x_3^{(n)} = x_3^*$. So there is the limit $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$. Thus $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = (x_1^*, 0, 1 - x_1^*)$ for any $\mathbf{x}^{(0)} \in S^2$ with $x_2^{(0)} > 0$.

We have proved the following Theorem.

Theorem 2. *For any $\mathbf{x}^{(0)} \in S^2$ the set limit points of trajectories of QSO defined by* (3.3) *and* (3.7) *has the form:*

$$\omega\big(\mathbf{x}^{(0)}\big) = \begin{cases} \{\big(x_1^{(0)}, 0, 1 - x_1^{(0)}\big)\}, & \text{if } \beta = 0,\ \gamma \neq 0, \\ \{\big(0, 1 - x_3^{(0)}, x_3^{(0)}\big)\}, & \text{if } \beta > 0,\ \gamma = 0, \\ \{\big(1 - x_3^{(0)}, 0, x_3^{(0)}\big)\}, & \text{if } \beta < 0,\ \gamma = 0, \\ \{\mathbf{x}^*\}, & \text{if } \beta\gamma \neq 0. \end{cases}$$

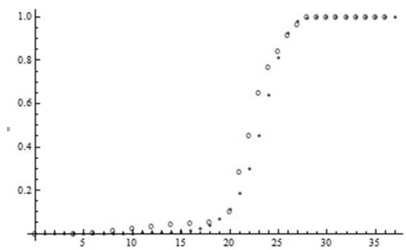

Figure 2. SIR model of Code Red propagation (black color points) and actual Code Red propagation (white color points).

Remark 4. Figure 2 shows the (3.3) SIR model's trajectory of Code Red propagation (in black color points) when $N = 350000$, $\beta = 0.75$, $I = 1$, (see [5]) $\gamma = 0.01$, $n = 36$ hours and actual Code Red propagation (in white color points).

3.3 Discrete time SIR model with the number of nodes is a variable

The discrete version of (2.3) has the following form

$$\begin{cases} s_{n+1} = -\beta s_n i_n - (\alpha + \gamma) s_n + \alpha, \\ i_{n+1} = \beta s_n i_n - (\alpha + \gamma) i_n, \\ r_{n+1} = \gamma (s_n + i_n) - \alpha r_n, \end{cases}$$

where n is a non-negative integer number.

Consider an operator $V : \mathbb{R}^3_+ \to \mathbb{R}^3_+$, $\mathbf{x} = (x_1, x_2, x_3) \to \mathbf{x}' = V(\mathbf{x}) = (x'_1, x'_2, x'_3)$ defined by

$$V_3 : \begin{cases} x'_1 = x_1(1 - \beta x_2 - \alpha - \gamma) + \alpha, \\ x'_2 = x_2(1 + \beta x_1 - \alpha - \gamma), \\ x'_3 = x_3(1 - \gamma - \alpha) + \gamma, \end{cases} \tag{3.8}$$

where $x_1' = s_{n+1}$, $x_2' = i_{n+1}$, $x_3' = r_{n+1}$, $x_1 = s_n$, $x_2 = i_n$, $x_3 = r_n$ for a natural n. Using $x_1 + x_2 + x_3 = 1$ we rewrite the operator (3.8) in the form

$$\begin{cases} x_1' = (1-\gamma)x_1^2 + \alpha x_2^2 + \alpha x_3^2 + (1-\beta-\gamma+\alpha)x_1x_2 + (1-\gamma+\alpha)x_1x_3 \\ \qquad +2\alpha x_2x_3, \\ x_2' = (1-\gamma-\alpha)x_2^2 + (1-\gamma-\alpha+\beta)x_1x_2 + (1-\gamma-\alpha)x_2x_3, \\ x_3'=\gamma x_1^2+\gamma x_2^2+(1-\alpha)x_3^2+2\gamma x_1x_2 + (1+\gamma-\alpha)x_1x_3 + (1+\gamma-\alpha)x_2x_3. \end{cases} \tag{3.9}$$

From the Equations (3.9) and (3.5) we have the following relations:

$$\begin{array}{lll} p_{11,1} =1-\gamma, & 2p_{12,1} =1-\beta-\gamma+\alpha, & 2p_{13,1} =1-\gamma+\alpha, \\ p_{11,2} =0, & 2p_{12,2} =1-\gamma-\alpha+\beta, & 2p_{13,2} =0, \\ p_{11,3} =\gamma, & 2p_{12,3} =2\gamma, & 2p_{13,3} =1+\gamma-\alpha, \\ p_{22,1} =\alpha, & 2p_{23,1} =2\alpha, & p_{33,1} =\alpha, \\ p_{22,2} =1-\gamma-\alpha, & 2p_{23,2} =1-\gamma-\alpha, & p_{33,2} =0, \\ p_{22,3} =\gamma, & 2p_{23,3} =1+\gamma-\alpha, & p_{33,3} =1-\alpha. \end{array}$$

Proposition 2. *The operator* (3.9) *maps* S^2 *to itself if and only if*

$$0 \le \gamma \le 1, \quad 0 \le \alpha \le 1, \quad \alpha+\gamma-1 \le \beta \le 1-\gamma+\alpha. \tag{3.10}$$

Proof. The proof is similar to proof of Proposition 1. □

Note that under condition (3.10) the operator V_3 is a non-Volterra QSO which is different from a $l-$Volterra QSO. Clearly that if $\alpha=\beta=\gamma=0$ then V_3 identity map.

I. a) Case $\alpha=\gamma=0$, $\beta>0$: In this case, one has that the points of the sets $\Gamma_{\{1\}} = \{\mathbf{x} \in S^2 : x_1 = 0\}$ and $\Gamma_{\{2\}} = \{\mathbf{x} \in S^2 : x_2 = 0\}$ are fixed points of operator V_3. Further, we get

$$x_1' = x_1(1-\beta x_2) \le x_1, \quad x_2' = x_2(1+\beta x_1) \ge x_2, \quad x_3' = x_3.$$

Therefore, it follows that

$$x_1^{(n)} \le x_1^{(n-1)}, \quad x_2^{(n)} \ge x_2^{(n-1)}, \quad x_3^{(n)} = x_3^{(0)}, \quad n = 1, 2, \dots$$

and it follows the existence of limits $\lim\limits_{n\to\infty} x_1^{(n)} = x_1^*$, $\lim\limits_{n\to\infty} x_2^{(n)} = x_2^*$ and $\lim\limits_{n\to\infty} x_3^{(n)} = x_3^* = x_3^{(0)}$. So, there is the limit $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$. Since the limit point $\mathbf{x}^*$ is a fixed point we have $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \left(0, 1-x_3^{(0)}, x_3^{(0)}\right)$ for any $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{2\}}$.

b) Case $\alpha=\gamma=0$, $\beta<0$: In this case, one has that the points of the sets $\Gamma_{\{1\}} = \{\mathbf{x} \in S^2 : x_1 = 0\}$ and $\Gamma_{\{2\}} = \{\mathbf{x} \in S^2 : x_2 = 0\}$ are fixed points of operator V_3. Further, we get

$$x_1' = x_1(1-\beta x_2) \ge x_1, \quad x_2' = x_2(1+\beta x_1) \le x_2, \quad x_3' = x_3.$$

Therefore, it follows that

$$x_1^{(n)} \geq x_1^{(n-1)}, \quad x_2^{(n)} \leq x_2^{(n-1)}, \quad x_3^{(n)} = x_3^{(0)}, \quad n = 1, 2, \ldots$$

and it follows the existence of limits $\lim_{n\to\infty} x_1^{(n)} = x_1^*$, $\lim_{n\to\infty} x_2^{(n)} = x_2^*$ and $\lim_{n\to\infty} x_3^{(n)} = x_3^* = x_3^{(0)}$. So, there is the limit $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$. Since the limit point $\mathbf{x}^*$ is a fixed point we have $\lim_{n\to\infty} \mathbf{x}^{(n)} = \big(1 - x_3^{(0)}, 0, x_3^{(0)}\big)$ for any $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1\}}$.

II. Case $\alpha = \beta = 0$, $\gamma \neq 0$: In this case, one has that the vertex $\mathbf{e}_3 = (0, 0, 1)$ is a unique fixed point of operator V_3. Further, we get

$$x_1' = x_1(1-\gamma) \leq x_1, \quad x_2' = x_2(1-\gamma) \leq x_2, \quad x_3' = x_3 + \gamma(1 - x_3).$$

Therefore, it follows

$$\lim_{n\to\infty} x_i^{(n+1)} = \lim_{n\to\infty} (1-\gamma) x_i^{(n)} = \lim_{n\to\infty} (1-\gamma)^{n+1} x_i^{(0)}, \quad i = 1, 2.$$

Consequently, there is $\lim_{n\to\infty} x_3^{(n)} = 1$. Thus $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} \in S^2$.

III. Case $\beta = \gamma = 0$, $\alpha \neq 0$: In this case, one has that the vertex $\mathbf{e}_1 = (1, 0, 0)$ is a unique fixed points of operator V_3. Further, we get

$$x_1' = x_1 + \alpha(1 - x_1) \geq x_1, \quad x_2' = x_2(1-\alpha) \leq x_2, \quad x_3' = x_3(1-\alpha) \leq x_3.$$

Therefore, it follows

$$\lim_{n\to\infty} x_i^{(n+1)} = \lim_{n\to\infty} (1-\alpha) x_i^{(n)} = \lim_{n\to\infty} (1-\alpha)^{n+1} x_i^{(0)}, \quad i = 2, 3.$$

Hence, there is $\lim_{n\to\infty} x_1^{(n)} = 1$. Thus, we have $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_1$ for any $\mathbf{x}^{(0)} \in S^2$.

IV. Case $\alpha = 0$, $\beta > 0$, $\gamma \neq 0$: In this case, one has that the vertex $\mathbf{e}_3 = (0, 0, 1)$ is a unique fixed points of operator V_3. Further, we get

$$x_1' = x_1(1 - \beta x_2 - \gamma) \leq x_1, \quad x_2' = x_2(1 + \beta x_1 - \gamma), \quad x_3' = (1-\gamma) x_3 + \gamma.$$

Since the third equation of (3.8) is a linear transformation, it follows that

$$\lim_{n\to\infty} x_3^{(n)} = \lim_{n\to\infty} \left((1-\gamma)^n x_3^{(0)} + \gamma \sum_{j=0}^{n-1} (1-\gamma)^j \right) = 1.$$

Consequently, there exists $\lim_{n\to\infty} \big(x_1^{(n)} + x_2^{(n)}\big) = 0$, i.e., $\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} x_2^{(n)} = 0$. Thus $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} \in S^2$.

The same result can be obtained in the case $\alpha = 0$, $\beta < 0$, $\gamma \neq 0$.

V. Case $\beta = 0$, $\alpha\gamma \neq 0$: In this case, one has that the point $\widetilde{\mathbf{x}} = (\alpha/(\alpha + \gamma), 0, \gamma/(\alpha + \gamma))$ is a unique fixed point of operator V_3. If $\alpha + \gamma = 1$, then $V_3(\mathbf{x}) = (\alpha, 0, \gamma)$ for any $\mathbf{x} \in S^2$.

Suppose $\alpha+\gamma\neq 1$, then we have

$$x_1'=x_1(1-\gamma-\alpha)+\alpha,\quad x_2'=x_2(1-\gamma-\alpha),\quad x_3'=x_3(1-\gamma-\alpha)+\gamma.$$

As these equations define a linear transformation it follows there are the limits

$$\lim_{n\to\infty}x_1^{(n)}=\lim_{n\to\infty}\Big((1-\gamma-\alpha)^n x_1^{(0)}+\alpha\sum_{j=0}^{n-1}(1-\gamma-\alpha)^j\Big)=\widetilde{x}_1=\frac{\alpha}{\alpha+\gamma},$$
$$\lim_{n\to\infty}x_2^{(n)}=\lim_{n\to\infty}(1-\gamma-\alpha)x_2^{(n-1)}=\lim_{n\to\infty}(1-\gamma-\alpha)^n x_2^{(0)}=0,$$
$$\lim_{n\to\infty}x_3^{(n)}=\lim_{n\to\infty}\Big((1-\gamma-\alpha)^n x_3^{(0)}+\gamma\sum_{j=0}^{n-1}(1-\gamma-\alpha)^j\Big)=\widetilde{x}_3=\frac{\gamma}{\alpha+\gamma}.$$

Thus, $\lim\limits_{n\to\infty}\mathbf{x}^{(n)}=\widetilde{\mathbf{x}}$ for any $\mathbf{x}^{(0)}\in S^2$.

VI. a) Case $\gamma=0$, $\beta<0$, $\alpha\neq 0$: In this case, one has that the vertex $\mathbf{e}_1=(1,0,0)$ is the unique fixed point of operator V_3. We have

$$x_1'=x_1(1-\beta x_2-\alpha)+\alpha,\quad x_2'=x_2(1+\beta x_1-\alpha)\le x_2,\quad x_3'=(1-\alpha)x_3\le x_3.$$

One has that $\lim\limits_{n\to\infty}x_2^{(n)}=\lim\limits_{n\to\infty}x_3^{(n)}=0$, i.e., $\lim\limits_{n\to\infty}\mathbf{x}^{(n)}=\mathbf{e}_1$ for any $\mathbf{x}^{(0)}\in S^2$.

b) Case $\gamma=0$, $\beta>0$, $\alpha\neq 0$: In this case, one has that the fixed points of operator V_3 are the vertex $\mathbf{e}_1=(1,0,0)$ and $\widehat{\mathbf{x}}=(\alpha/\beta,(\beta-\alpha)/\beta,0)$ whenever $\alpha\le\beta\le 1+\alpha$. We have

$$x_1'=x_1(1-\beta x_2-\alpha)+\alpha,\quad x_2'=x_2(1+\beta x_1-\alpha),\quad x_3'=(1-\alpha)x_3\le x_3.$$

If $\beta\le\alpha$ then $\lim\limits_{n\to\infty}\mathbf{x}^{(n)}=\mathbf{e}_1$, $\forall\mathbf{x}^{(0)}\in S^2$ and if $\beta>\alpha$ then $\lim\limits_{n\to\infty}\mathbf{x}^{(n)}=\widehat{\mathbf{x}}$ for any $\mathbf{x}^{(0)}\in S^2$.

The proof of these claims can be obtained from the results of the next case setting $\gamma=0$.

VII. a) Case $\alpha\gamma>0$, $\beta>0$: In this case, one has that the third equation of (3.8) is a linear transformation. It follows that

$$\lim_{n\to\infty}x_3^{(n)}=\lim_{n\to\infty}\Big((1-\gamma-\alpha)^n x_3^{(0)}+\gamma\sum_{j=0}^{n-1}(1-\gamma-\alpha)^j\Big)=x_3^*=\frac{\gamma}{\alpha+\gamma}.$$

Evidently, the set $\mathit{\Gamma}_{\{2\}}$ is an invariant. So, for any $\mathbf{x}^{(0)}\in\mathit{\Gamma}_{\{2\}}$ we have

$$\lim_{n\to\infty}\mathbf{x}^{(n)}=\Big(\frac{\alpha}{\alpha+\gamma},0,\frac{\gamma}{\alpha+\gamma}\Big).$$

Let $\beta\le\alpha$, then $x_2'\le x_2$, consequently $x_2^{(n)}\le x_2^{(n-1)}$, that is $\{x_2^{(n)}\}$ is a decreasing and bounded from the below sequence. Consequently, there exists $\lim\limits_{n\to\infty}x_2^{(n)}=x_2^*$ and it follows existence of $\lim\limits_{n\to\infty}x_1^{(n)}=x_1^*=1-x_2^*-x_3^*$. Therefore, $\lim\limits_{n\to\infty}\mathbf{x}^{(n)}=\mathbf{x}^*$ for any $\mathbf{x}^{(0)}\in S^2$. Since the points

$$\mathbf{y}=\Big(\frac{\alpha}{\alpha+\gamma},0,\frac{\gamma}{\alpha+\gamma}\Big),\quad \mathbf{z}=\Big(\frac{\alpha+\gamma}{\beta},\frac{\alpha\beta-(\alpha+\gamma)^2}{(\alpha+\gamma)\beta},\frac{\gamma}{\alpha+\gamma}\Big)$$

are fixed points of operator V_3 and the limit point $\mathbf{x}^*$ should be a fixed point we get either $\mathbf{x}^* = \mathbf{y}$ or $\mathbf{x}^* = \mathbf{z}$. But using $0 < \beta \le \alpha$, one has

$$\alpha\beta - (\alpha+\gamma)^2/(\alpha+\gamma)\beta < 0.$$

So, the point $\mathbf{z}$ cannot be a fixed point and we get $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{y}$ for a $\mathbf{x}^{(0)} \in S^2$.

Let $\beta > \alpha$. Then from $\lim\limits_{n\to\infty} x_3^{(n)} = x_3^*$ it follows that for any $\varepsilon > 0$, there is a natural n_0 such that for all $n > n_0$ it holds $x_3^* - \varepsilon < x_3^{(n)} < x_3^* + \varepsilon$. From the second equation of (3.8), we have

$$\varphi\big(x_2^{(n)}\big) - \beta\varepsilon < x_2^{(n+1)} < \varphi\big(x_2^{(n)}\big) + \beta\varepsilon,$$

where $\varphi(x) = x\big(\beta(1-x) + 1 - \gamma - \alpha - \beta x_3^*\big)$.

Definition 3. Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are called topologically conjugate if there exists a homeomorphism $h : A \to B$ such that, $h \circ f = g \circ h$.

Note that mappings which are topologically conjugate are completely equivalent in terms of their dynamics. In particular, h gives a one-to-one correspondence between the set of limit points of f and g.

Consider the function $\varphi(x)$. Taking $h(x) = ax + b$ one can see that the function $f(x)$ is topologically conjugate to the well-known logistic map $g(x) = \mu x(1-x)$ with $\mu = 1 + \alpha\beta - (\alpha+\gamma)^2/(\alpha+\gamma)$.

Using $\beta > \alpha$ and (3.7) we have $\mu \in (0;3)$. For the logistic map the following is known (see [8]): (i) if $\mu \in (0,1]$ then g has an attracting unique fixed point $x = 0$. The all trajectories will approach fixed point $x = 0$; (ii) if $\mu \in (1,3)$ then g has an attracting fixed point $\widetilde{x} = (\mu-1)/\mu$ and repelling fixed point $x = 0$. The all trajectories (except when started at the fixed point) will approach to fixed point $\widetilde{x}$.

From this fact, by the conjugacy argument, it follows that if $\alpha\beta \le (\alpha+\gamma)^2$ then all trajectories of $\varphi(x)$ started an initial point will approach fixed point $x = 0$. If $\alpha\beta > (\alpha+\gamma)^2$ then all trajectories of $\varphi(x)$ started an initial point will approach fixed point

$$\widehat{x} = \big(\alpha\beta - (\alpha+\gamma)^2\big)/((\alpha+\gamma)\beta).$$

Thus $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{z}$ for any $\mathbf{x}^{(0)} \in S^2 \setminus \Big(\mathit{\Gamma}_{\{1\}} \cup \mathit{\Gamma}_{\{2\}}\Big)$.

b) Case $\alpha\gamma > 0$, $\beta < 0$: In this case, one has that the third equation of (3.8) is a linear transformation. It follows that

$$\lim_{n\to\infty} x_3^{(n)} = \lim_{n\to\infty} \Big((1-\gamma-\alpha)^n x_3^{(0)} + \gamma\sum_{j=0}^{n-1}(1-\gamma-\alpha)^j\Big) = x_3^* = \frac{\gamma}{\alpha+\gamma}.$$

As in the above case the set $\mathit{\Gamma}_{\{2\}}$ is an invariant and for a $\mathbf{x}^{(0)} \in \mathit{\Gamma}_{\{2\}}$, we have

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \Big(\frac{\alpha}{\alpha+\gamma}, 0, \frac{\gamma}{\alpha+\gamma}\Big).$$

It is clear that $x_2' \leq x_2$, consequently $x_2^{(n)} \leq x_2^{(n-1)}$, that is $\{x_2^{(n)}\}$ is a decreasing and bounded from the below sequence. Consequently, there exists $\lim\limits_{n\to\infty} x_2^{(n)} = x_2^*$ and it follows existence of $\lim\limits_{n\to\infty} x_1^{(n)} = x_1^* = 1 - x_2^* - x_3^*$. Therefore, $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$ for any $\mathbf{x}^{(0)} \in S^2$.

Since the point $\mathbf{y} = \left(\frac{\alpha}{\alpha+\gamma}, 0, \frac{\gamma}{\alpha+\gamma}\right)$ is a fixed points of operator V_3 and the limit point $\mathbf{x}^*$ should be a fixed point, so we get $\mathbf{x}^* = \mathbf{y}$. Thus, we obtain $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{y}$ for any $\mathbf{x}^{(0)} \in S^2 \setminus \left(\Gamma_{\{1\}} \cup \Gamma_{\{2\}}\right)$.

Thus, independently on parameters α, β, γ and initial point $\mathbf{x}^{(0)} \in S^2$ the limit of trajectory exists. Summarizing the results of cases **I–VII**, we obtain

Theorem 3. *For any* $\mathbf{x}^{(0)} \in S^2$ *and for the QSO* (3.8) *with parameters* (3.10) *we have*

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \begin{cases} \left(0, 1 - x_3^{(0)}, x_3^{(0)}\right), & \text{if } \alpha = \gamma = 0,\ \beta > 0, \\ \left(1 - x_3^{(0)}, 0, x_3^{(0)}\right), & \text{if } \alpha = \gamma = 0,\ \beta < 0, \\ \mathbf{e}_3, & \text{if } \alpha = \beta = 0,\ \gamma \neq 0, \\ \mathbf{e}_1, & \text{if } \beta = \gamma = 0,\ \alpha \neq 0, \\ \mathbf{e}_3, & \text{if } \alpha = 0,\ \beta\gamma \neq 0, \\ \widetilde{\mathbf{x}}, & \text{if } \beta = 0,\ \alpha\gamma \neq 0, \\ \mathbf{e}_1, & \text{if } \gamma = 0,\ \alpha\beta \neq 0, \beta \leq \alpha, \\ \widehat{\mathbf{x}}, & \text{if } \gamma = 0,\ \alpha\beta \neq 0, \beta > \alpha, \\ \mathbf{y}, & \text{if } \alpha\beta\gamma \neq 0,\ \beta \leq \alpha, \\ \mathbf{y}, & \text{if } \alpha\beta\gamma \neq 0,\ \beta > \alpha, \alpha\beta \leq (\alpha+\gamma)^2, \\ \mathbf{z}, & \text{if } \alpha\beta\gamma \neq 0,\ \beta > \alpha, \alpha\beta > (\alpha+\gamma)^2. \end{cases}$$

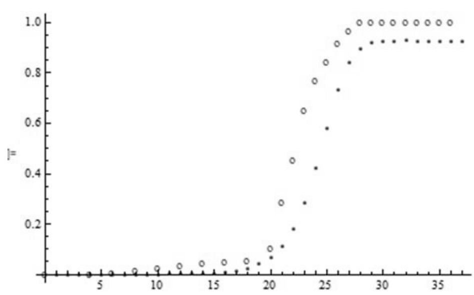

Figure 3. SIR model of Code Red propagation (black color points) and actual Code Red propagation (white color points).

Remark 5. Figure 3 shows the (3.8) SIR model's trajectory of Code Red propagation (in black color points) when $N = 350000$, $\beta = 0.75$, $I = 1$ (see [5]), $\gamma = 0.001$, $\alpha = 0.4$, $n = 36$ hours and actual Code Red propagation (in white color points).

4 Discrete time PSIDR model

Let us consider discrete time analogues of the PSIDR models of worm propagation. As in the above we make some relations with $p_{ij,k}$ we find conditions on parameters of models (operators) rewriting them in the form (2.6).

Let us fix a natural number t and for any natural n the discrete versions of the models (2.4), (2.5) has the following form

$$\begin{cases} s_{n+1} = s_n - \beta s_n i_n - \mathbf{1}_{\{n\geq t\}}\mu s_n, \\ i_{n+1} = i_n + \beta s_n i_n - \mathbf{1}_{\{n\geq t\}}\mu i_n, \\ d_{n+1} = \mathbf{1}_{\{n\geq t\}}\left((1-\delta)d_n + \mu i_n\right), \\ r_{n+1} = \mathbf{1}_{\{n\geq t\}}\left(r_n + \delta d_n + \mu s_n\right), \end{cases}$$

where n, m are non-negative integer numbers and $\mathbf{1}_{\{n\geq t\}} = 0$ (resp. $= 1$) if $n < t$ (resp. $n \geq t$).

Consider an operator $V : \mathbb{R}^4_+ \to \mathbb{R}^4_+$, $\mathbf{x}^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}\right) \to \mathbf{x}^{(n+1)} = V(\mathbf{x}) = \left(x_1^{(n+1)}, x_2^{(n+1)}, x_3^{(n+1)}, x_4^{(n+1)}\right)$ defined by

$$V_4 : \begin{cases} x_1^{(n+1)} = x_1^{(n)}(1 - \beta x_2^{(n)} - \mathbf{1}_{\{n\geq t\}}\mu), \\ x_2^{(n+1)} = x_2^{(n)}(1 + \beta x_1^{(n)} - \mathbf{1}_{\{n\geq t\}}\mu), \\ x_3^{(n+1)} = \mathbf{1}_{\{n\geq t\}}(x_3^{(n)}(1-\delta) + \mu x_2^{(n)}), \\ x_4^{(n+1)} = \mathbf{1}_{\{n\geq t\}}(x_4^{(n)} + \delta x_3^{(n)} + \mu x_1), \end{cases}$$

where $x_1^{(n+1)} = s_{n+1}$, $x_2^{(n+1)} = i_{n+1}$, $x_3^{(n+1)} = d_{n+1}$, $x_4^{(n+1)} = r_{n+1}$, $x_1^{(n)} = s_n$, $x_2^{(n)} = i_n$, $x_3^{(n)} = d_n$, $x_4^{(n)} = r_n$. Using the notation $x_1' = x_1^{(n+1)}$, $x_2' = x_2^{(n+1)}$, $x_3' = x_3^{(n+1)}$, $x_4' = x_4^{(n+1)}$, $x_1 = x_1^{(n)}$, $x_2 = x_2^{(n)}$, $x_3 = x_3^{(n)}$, $x_4 = x_4^{(n)}$ we have

$$V_4 : \begin{cases} x_1' = x_1(1 - \beta x_2 - \mathbf{1}_{\{n\geq t\}}\mu), \\ x_2' = x_2(1 + \beta x_1 - \mathbf{1}_{\{n\geq t\}}\mu), \\ x_3' = \mathbf{1}_{\{n\geq t\}}(x_3(1-\delta) + \mu x_2), \\ x_4' = \mathbf{1}_{\{n\geq t\}}(x_4 + \delta x_3 + \mu x_1). \end{cases} \tag{4.1}$$

For $n = 0, \ldots, m-1$ the operator V_4 coincides with V_1 (see (3.1)) and when $n \geq t$ using $x_1 + x_2 + x_3 + x_4 = 1$ we rewrite the operator (4.1) in the form

$$\begin{cases} x_1' = (1-\mu)x_1^2 + (1-\beta-\mu)x_1x_2 + (1-\mu)x_1x_3 + (1-\mu)x_1x_4, \\ x_2' = (1-\mu)x_2^2 + (1+\beta-\mu)x_1x_2 + (1-\mu)x_2x_3 + (1-\mu)x_2x_4, \\ x_3' = \mu x_2^2 + (1-\delta)x_3^2 + \mu x_1x_2 + (1-\delta)x_1x_3 + (1-\delta+\mu)x_2x_3 \\ \qquad +\mu x_2x_4 + (1-\delta+\mu)x_3x_4, \\ x_4' = \mu x_1^2 + \delta x_3^2 + x_4^2 + \mu x_1x_2 + (\mu+\delta)x_1x_3 + (1+\mu)x_1x_4 \\ \qquad +\delta x_2x_3 + x_2x_4 + (1+\delta)x_3x_4. \end{cases} \tag{4.2}$$

From the Eqs. (2.6), when m=4 and (4.2), we have the following relations:

$$p_{11,1} = 1-\mu, \quad p_{22,1} = 0, \quad p_{33,1} = 0, \quad p_{44,1} = 0,$$

$$
\begin{array}{llll}
p_{11,2} =0, & p_{22,2} =1-\mu, & p_{33,2} =0, & p_{44,2} =0, \\
p_{11,3} =0, & p_{22,3} =\mu, & p_{33,3} =1-\delta, & p_{44,3} =0, \\
p_{11,4} =\mu, & p_{22,4} =0, & p_{33,4} =\delta, & p_{44,4} =1. \\
2p_{12,1} =1-\beta-\mu, & 2p_{13,1} =1-\mu, & 2p_{14,1} =1-\mu, & \\
2p_{12,2} =1+\beta-\mu, & 2p_{13,2} =0, & 2p_{14,2} =0, & \\
2p_{12,3} =\mu, & 2p_{13,3} =1-\delta, & 2p_{14,3} =0, & \\
2p_{12,4} =\mu, & 2p_{13,4} =\mu+\delta, & 2p_{14,4} =1+\mu, & \\
2p_{23,1} =0, & 2p_{24,1} =0, & 2p_{34,1} =0, & \\
2p_{23,2} =1-\mu, & 2p_{24,2} =1-\mu, & 2p_{34,2} =0, & \\
2p_{23,3} =1+\mu-\delta, & 2p_{34,3} =1-\delta, & 2p_{14,3} =0, & \\
2p_{23,4} =\delta, & 2p_{24,4} =1, & 2p_{34,4} =1+\delta, &
\end{array}
$$

Proposition 3. *The operator* (4.2) *maps* S^3 *to itself if and only if*

$$0 \leq \mu \leq 1, \quad 0 \leq \delta \leq 1, \quad \mu - 1 \leq \beta \leq 1 - \mu. \tag{4.3}$$

Proof. The proof is similar to proof of Proposition 1. □

Note under condition (4.3) the operator (4.2) is a non-Volterra QSO. It is clear that if $\mu = \delta = \beta = 0$, then the operator (4.1) is the identity map.

Theorem 4. *For any* $\mathbf{x}^{(0)} \in S^3$ *the set* $\omega\big(\mathbf{x}^{(0)}\big)$ *of trajectories of QSO defined by* (4.1) *and* (4.3) *has the form:*

$$
\omega\big(\mathbf{x}^{(0)}\big) = \begin{cases}
\{\big(0, 1 - x_3^{(0)} - x_4^{(0)}, x_3^{(0)}, x_4^{(0)}\big)\}, & \textit{if } \ \mu = \delta = 0, \ \beta > 0, \\
\{\big(1 - x_3^{(0)} - x_4^{(0)}, 0, x_3^{(0)}, x_4^{(0)}\big)\}, & \textit{if } \ \mu = \delta = 0, \ \beta < 0, \\
\{\big(x_1^{(0)}, x_2^{(0)}, 0, 1 - x_1^{(0)} - x_2^{(0)}\big)\}, & \textit{if } \ \mu = \beta = 0, \ \delta \neq 0, \\
\{\big(0, 0, x_3^*, 1 - x_3^*\big)\}, & \textit{if } \ \beta = \delta = 0, \ \mu \neq 0, \\
\{\big(0, x_2^*, 0, 1 - x_2^*\big)\}, & \textit{if } \ \mu = 0, \ \delta \neq 0, \beta > 0, \\
\{\big(x_1^*, 0, 0, 1 - x_1^*\big)\}, & \textit{if } \ \mu = 0, \ \delta \neq 0, \beta < 0, \\
\{\big(0, 0, \widehat{x}_3, 1 - \widehat{x}_3\big)\}, & \textit{if } \ \delta = 0, \ \mu\beta \neq 0 \\
\{\mathbf{e}_4\}, & \textit{if } \ \mu\delta \neq 0.
\end{cases}
$$

Proof. We consider all possible cases.
I. a) Case $\mu = \delta = 0$, $\beta > 0$: In this case, one has that the points of the sets $\Gamma_{\{1\}} = \{\mathbf{x} \in S^3 : x_1 = 0\}$ and $\Gamma_{\{2\}} = \{\mathbf{x} \in S^3 : x_2 = 0\}$ are fixed points of operator V_4. Further, we get

$$x_1' = x_1(1 - \beta x_2) \leq x_1, \; x_2' = x_2(1 + \beta x_1) \geq x_2, \; x_3' = x_3, \; x_4' = x_4.$$

It is clear that $\lim\limits_{n\to\infty} x_i^{(n)} = x_i^* = x_i^{(0)}, \quad i = 3, 4$. Also, it follows that $x_1^{(n)} \leq x_1^{(n-1)}$ and $x_2^{(n)} \geq x_2^{(n-1)}$ for $n = 1, 2, \ldots$. As $\{x_1^{(n)}\}$ is a bounded from the below decreasing sequence and $\{x_2^{(n)}\}$ is a bounded from the above increasing sequence, we obtain there exist $\lim\limits_{n\to\infty} x_1^{(n)} = x_1^*$ and $\lim\limits_{n\to\infty} x_2^{(n)} = x_2^*$. Since the limit point $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ should be a fixed point, it follows that $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \big(0, 1 - x_3^{(0)} - x_4^{(0)}, x_3^{(0)}, x_4^{(0)}\big)$ for any $\mathbf{x}^{(0)} \in S^3$ with $x_2^{(0)} > 0$.

b) Case $\mu = \delta = 0$, $\beta < 0$: In this case, also one has that the points of the sets $\Gamma_{\{1\}} = \{\mathbf{x} \in S^3 : x_1 = 0\}$ and $\Gamma_{\{2\}} = \{\mathbf{x} \in S^3 : x_2 = 0\}$ are fixed points of operator V_4. Further, we get

$$x_1' = x_1(1 - \beta x_2) \geq x_1,\ x_2' = x_2(1 + \beta x_1) \leq x_2,\ x_3' = x_3,\ x_4' = x_4.$$

It is clear that $\lim\limits_{n\to\infty} x_i^{(n)} = x_i^* = x_i^{(0)}, \quad i = 3, 4$. Also, it follows that $x_1^{(n)} \leq x_1^{(n-1)}$ and $x_2^{(n)} \geq x_2^{(n-1)}$ for $n = 1, 2, \ldots$.

As $\{x_1^{(n)}\}$ is a bounded from the above increasing sequence and $\{x_2^{(n)}\}$ is a bounded from below decreasing the sequence we obtain there exist $\lim\limits_{n\to\infty} x_1^{(n)} = x_1^*$ and $\lim\limits_{n\to\infty} x_2^{(n)} = x_2^*$. Since the limit point $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ should be a fixed point, it follows that $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \big(1 - x_3^{(0)} - x_4^{(0)}, 0, x_3^{(0)}, x_4^{(0)}\big)$ for any $\mathbf{x}^{(0)} \in S^3$ with $x_1^{(0)} > 0$. Thus we have proved following Proposition.

Proposition 4. *Let $\mu = \delta = 0$ then for any $\mathbf{x}^{(0)} \in S^3$ it holds*

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \begin{cases} \big(0, 1 - x_3^{(0)} - x_4^{(0)}, x_3^{(0)}, x_4^{(0)}\big), & \text{if } \beta > 0, \\ \big(1 - x_3^{(0)} - x_4^{(0)}, 0, x_3^{(0)}, x_4^{(0)}\big), & \text{if } \beta < 0. \end{cases}$$

II. Case $\mu = \beta = 0$, $\delta \neq 0$: In this case, one has that the points of the set $\Gamma_{\{3\}} = \{\mathbf{x} \in S^3 : x_3 = 0\}$ are fixed points of operator V_4. We have

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = (1 - \delta)x_3, \quad x_4' = x_4 + \delta x_3.$$

So, $\lim\limits_{n\to\infty} x_i^{(n)} = x_i^{(0)}$, $i = 1, 2$ and it follows $x_4^{(n)} \geq x_4^{(n-1)}$ for $n = 1, 2, \ldots$.

As $\{x_4^{(n)}\}$ is an increasing and bounded from the above sequence one has there exists $\lim\limits_{n\to\infty} x_4^{(n)} = x_4^*$. Further,

$$\lim_{n\to\infty} x_3^{(n)} = \lim_{n\to\infty} (1 - \delta) x_3^{(n-1)} = \lim_{n\to\infty} (1 - \delta)^n x_3^{(0)} = 0.$$

Since the limit point $\mathbf{x}^* = (x_1^{(0)}, x_2^{(0)}, 0, x_4^*)$ should be a fixed point, it follows that $x_4^* = 1 - x_1^{(0)} - x_2^{(0)}$, that is $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = (x_1^{(0)}, x_2^{(0)}, 0, 1 - x_1^{(0)} - x_2^{(0)})$ for any $\mathbf{x}^{(0)} \in S^3$ with $x_3^{(0)} > 0$. Thus, we have proved following Proposition.

Proposition 5. *If $\mu = \beta = 0$, $\delta \neq 0$, then $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = (x_1^{(0)}, x_2^{(0)}, 0, 1 - x_1^{(0)} - x_2^{(0)})$ for any $\mathbf{x}^{(0)} \in S^3$ with $x_3^{(0)} > 0$.*

III. Case $\beta = \delta = 0$, $\mu \neq 0$: In this case, it's easy to see that the points of the set $\Gamma_{\{12\}} = \{\mathbf{x} \in S^3 : x_1 = x_2 = 0\}$ are fixed points of operator V_4. One has

$$\begin{aligned} x_1' &= x_1(1 - \mu) \leq x_1, \quad & x_2' &= x_2(1 - \mu) \leq x_2, \\ x_3' &= x_3 + \mu x_2 \geq x_3, \quad & x_4' &= x_4 + \mu x_1 \geq x_4. \end{aligned}$$

Moreover, it follows

$$\lim_{n\to\infty} x_i^{(n)} = \lim_{n\to\infty}(1-\mu)x_i^{(n-1)} = \lim_{n\to\infty}(1-\mu)^n x_i^{(0)} = 0, \quad i = 1,2.$$

Also we get that the sequences $\{x_3^{(n)}\}$,$\{x_4^{(n)}\}$ are increasing and bounded from the above sequences. Therefore, there are the limits $\lim\limits_{n\to\infty} x_i^{(n)} = x_i^*, \;\; i = 3,4$. Since the limit point $\mathbf{x}^* = (0,0,x_3^*,x_4^*)$ should be a fixed point, it follows that $x_4^* = 1-x_3^*$, that is $\lim\limits_{n\to\infty}\mathbf{x}^{(n)} = (0,0,x_3^*,1-x_3^*)$ for any $\mathbf{x}^{(0)} \in S^3$ with either $x_1^{(0)} > 0$ or $x_2^{(0)} > 0$. Thus we have proved following Proposition.

Proposition 6. *If $\beta = \delta = 0$, $\mu \neq 0$, then $\lim\limits_{n\to\infty}\mathbf{x}^{(n)} = (0,0,x_3^*,1-x_3^*)$ for any $\mathbf{x}^{(0)} \in S^3$ with either $x_1^{(0)} > 0$ or $x_2^{(0)} > 0$.*

IV. a) Case $\mu = 0$, $\delta \neq 0$, $\beta > 0$: In this case, one has that the points of the sets $\Gamma_{\{13\}} = \{\mathbf{x} \in S^3 : x_1 = x_3 = 0\}$ and $\Gamma_{\{14\}} = \{\mathbf{x} \in S^3 : x_1 = x_4 = 0\}$ are fixed points of operator V_4. We have

$$\begin{aligned} x_1' &= x_1(1-\beta x_2) \le x_1, \quad x_2' = x_2(1+\beta x_1) \ge x_2,\\ x_3' &= x_3(1-\delta) \le x_3, \quad x_4' = x_4 + \delta x_3 \ge x_4. \end{aligned}$$

Therefore, it follows that $\{x_1^{(n)}\}$,$\{x_3^{(n)}\}$ are bounded from the below decreasing sequences and $\{x_2^{(n)}\}$,$\{x_4^{(n)}\}$ are bounded from the above increasing sequences. Consequently, there are the limits $\lim\limits_{n\to\infty} x_i^{(n)} = x_i^*, \;\; i = 1,2,3,4$. Since the limit point $\mathbf{x}^* = (x_1^*,x_2^*,x_3^*,x_4^*)$ should be a fixed point, it follows that $\lim\limits_{n\to\infty}\mathbf{x}^{(n)} = (0,x_2^*,0,1-x_2^*)$ for any $\mathbf{x}^{(0)} \in S^3 \setminus (\Gamma_{\{13\}} \cup \Gamma_{\{14\}})$.

b) Case $\mu = 0$, $\delta \neq 0$, $\beta < 0$: In this case, also the points of the sets $\Gamma_{\{13\}} = \{\mathbf{x} \in S^3 : x_1 = x_3 = 0\}$ and $\Gamma_{\{14\}} = \{\mathbf{x} \in S^3 : x_1 = x_4 = 0\}$ are fixed points of operator V_4. We have

$$\begin{aligned} x_1' &= x_1(1-\beta x_2) \ge x_1, \quad x_2' = x_2(1+\beta x_1) \le x_2,\\ x_3' &= x_3(1-\delta) \le x_3, \quad x_4' = x_4 + \delta x_3 \ge x_4. \end{aligned}$$

Therefore, it follows that $\{x_1^{(n)}\}$,$\{x_4^{(n)}\}$ are bounded from the above increasing sequences and $\{x_2^{(n)}\}$,$\{x_3^{(n)}\}$ are bounded from the below decreasing sequences. Consequently, there are the limits $\lim\limits_{n\to\infty} x_i^{(n)} = x_i^*, \;\; i = 1,2,3,4$. Since the limit point $\mathbf{x}^* = (x_1^*,x_2^*,x_3^*,x_4^*)$ should be a fixed point it follows that $\lim\limits_{n\to\infty}\mathbf{x}^{(n)} = (x_1^*,0,0,1-x_1^*)$ for any $\mathbf{x}^{(0)} \in S^3 \setminus (\Gamma_{\{13\}} \cup \Gamma_{\{14\}})$. Thus, we have

Proposition 7. *Let $\mu = 0$, $\delta \neq 0$, then for any $\mathbf{x}^{(0)} \in S^3$ it holds*

$$\lim_{n\to\infty}\mathbf{x}^{(n)} = \begin{cases} \left(0,x_2^*,0,1-x_2^*\right), & \text{if } \;\beta > 0,\\ \left(x_1^*,0,0,1-x_1^*\right), & \text{if } \;\beta < 0. \end{cases}$$

V. Case $\delta = 0$, $\mu \neq 0, \beta > 0$: In this case, one has that the points of the set $\mathit{\Gamma}_{\{12\}} = \{\mathbf{x} \in S^3 : x_1 = x_2 = 0\}$ are fixed points of operator V_4. We have

$$\begin{aligned} x_1' &= x_1(1 - \beta x_2 - \mu) \geq x_1, \quad x_2' = x_2(1 + \beta x_1 - \mu), \\ x_3' &= x_3 + \mu x_2 \geq x_3, \quad x_4' = x_4 + \mu x_1 \geq x_4. \end{aligned}$$

So, it follows that $\{x_3^{(n)}\}$,$\{x_4^{(n)}\}$ are bounded from the above increasing sequences. Hence, there are the limits $\lim\limits_{n\to\infty} x_i^{(n)} = \widehat{x}_i$, $i = 3, 4$. One has that

$$\lim_{n\to\infty} \big(x_1^{(n)}+x_2^{(n)}\big)= \lim_{n\to\infty} (1-\mu)\big(x_1^{(n-1)} + x_2^{(n-1)}\big)= \lim_{n\to\infty} (1-\mu)^n \big(x_1^{(0)}+x_2^{(0)}\big) = 0,$$

that is $\lim\limits_{n\to\infty} x_1^{(n)} = \lim\limits_{n\to\infty} = x_2^{(n)} = 0$. Thus, $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = (0, 0, \widehat{x}_3, 1 - \widehat{x}_3)$ for any $\mathbf{x}^{(0)} \in S^3 \setminus \mathit{\Gamma}_{\{12\}}$.

The same result can be obtained in the case $\delta = 0$, $\mu \neq 0, \beta < 0$. Thus, we have proved following Proposition.

Proposition 8. *If* $\delta = 0$, $\mu \neq 0$, *then* $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = (0, 0, \widehat{x}_3, 1 - \widehat{x}_3)$ *for any* $\mathbf{x}^{(0)} \in S^3 \setminus \mathit{\Gamma}_{\{12\}}$.

VI. a) Case $\mu\delta \neq 0$, $\beta = 0$: In this case, it is easy to verify that the vertex $\mathbf{e}_4 = (0, 0, 0, 1)$ is a unique fixed point of operator V_4. One has

$$\begin{aligned} x_1' &= x_1(1 - \mu) \leq x_1, \quad x_2' = x_2(1 - \mu) \leq x_2, \\ x_3' &= (1 - \delta)x_3 + \mu x_2, \quad x_4' = x_4 + \delta x_3 + \mu x_1 \geq x_4. \end{aligned}$$

Hence, $\{x_4^{(n)}\}$ is an increasing and bounded from the above sequence. Consequently, there is the limit $\lim\limits_{n\to\infty} x_4^{(n)} = x_4^*$. We have

$$\lim_{n\to\infty} x_i^{(n)} = \lim_{n\to\infty} (1 - \mu)x_i^{(n-1)} = \lim_{n\to\infty} (1 - \mu)^n x_i^{(0)} = 0, \quad i = 1, 2.$$

Hence, there is the limit $\lim\limits_{n\to\infty} x_3^{(n)} = 1-x_4^*$. Since the limit point $\mathbf{x}^* = (0, 0, 1-x_4^*, x_4^*)$ should be a fixed point, it follows that $x_4^* = 1$, that is $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_4$ for any $\mathbf{x}^{(0)} \in S^3$.

b) Case $\mu\delta \neq 0$, $\beta > 0$: In this case, it is easy to verify that the vertex $\mathbf{e}_4 = (0, 0, 0, 1)$ is a unique fixed point of operator V_4. One has

$$\begin{aligned} x_1' &= x_1(1 - \beta x_2 - \mu) \geq x_1, \quad x_2' = x_2(1 + \beta x_1 - \mu), \\ x_3' &= (1 - \delta)x_3 + \mu x_2, \quad x_4' = x_4 + \delta x_3 + \mu x_1 \geq x_4. \end{aligned}$$

Hence it follows that the sequence $\{x_4^{(n)}\}$ is a bounded from the above increasing sequence. Therefore there exits the limits $\lim\limits_{n\to\infty} x_4^{(n)} = x_4^*$. It easy to check that

$$\lim_{n\to\infty} \big(x_1^{(n)}+x_2^{(n)}\big)= \lim_{n\to\infty} (1-\mu)\big(x_1^{(n-1)}+x_2^{(n-1)}\big)= \lim_{n\to\infty} (1 - \mu)^n \big(x_1^{(0)}+x_2^{(0)}\big)=0,$$

that is $\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} = x_2^{(n)} = 0$. Consequently, there is the limit $\lim_{n\to\infty} x_3^{(n)} = 1 - x_4^*$. Since the limit point $\mathbf{x}^* = (0, 0, 1 - x_4^*, x_4^*)$ should be a fixed point, it follows that $x_4^* = 1$. Thus, $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_4$ for any $\mathbf{x}^{(0)} \in S^3$.

The same result can be obtained in the case $\mu\delta \neq 0$, $\beta < 0$.

Proposition 9. *If $\mu\delta \neq 0$, then $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_4$ for any $\mathbf{x}^{(0)} \in S^3$.*

Collecting together all Propositions 4–9 we finish the proof of Theorem 4. □

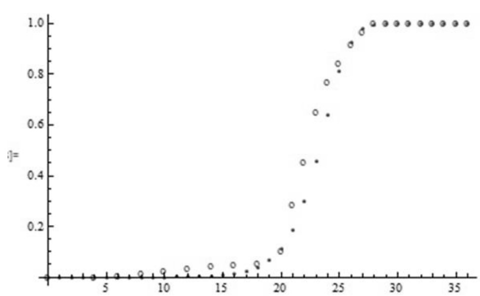

Figure 4. PSIDR model of Code Red propagation (black color points) and actual Code Red propagation (white color points)

Remark 6. Figure 4 shows the (4.1) PSIDR model‘s trajectory of Code Red propagation (in black color points), when $N = 350000$, $\beta = 0.75$, $I = 1$ (see [5]), $\mu = 0.0001$, $\delta = 0.2$, $n = 36$ hours, here 12 points of SIS model and the next 24 points of PSIDR model and actual Code Red propagation (in white color points).

5 Conclusions

We have considered discrete-time dynamical systems generated by known SI-SIR-PSIDR worm propagation models and studied their trajectory behaviours on the basis of the theory of QSOs. We showed that any trajectory of such models approaches to a fixed point (Theorems 1-4), that is the future of such system is stable (predictable).

It is worth mentioning that in [18] for the continuous time SIR model (2.3) is given a sufficient condition for worm propagation. In Theorem 3 for the discrete version of (2.3) the full description the set of limit points is given. The theoretically analysis of the continuous time PSIDR model (2.5) gives that formally for any practically realized parameters any worm no virus (single virus) is able to completely repress the computer network if $S(0) > 0$ [3]. The Theorem 4 shows that the worm theoretically can do it and it depend of the parameters and an initial state of network (e.g. when $\delta > 0$ enough small).

Note the cases when $\beta < 0$ which are impossible in real epidemic situations, but these cases are also interesting from a purely mathematical point of view. For example for the PSIDR models in another cases the obtained results have the following interpretations: Let $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\right) \in S^3$ be an initial state (point), that is $\mathbf{x}^{(0)}$ is the probability distribution on the set $\{S, I, D, R\}$ of system. By Theorem 4 the trajectory $\mathbf{x}^{(n)}$ of the initial point approaches a limit point $\mathbf{x}^*$ (an equilibrium state) this means that the future of the system is predictable: each S, I, D, R arises with probability $x_1^*, x_2^*, x_3^*, x_4^*$, respectively. For example, the infected nodes, I, of the total network (nodes) will disappear if its probability x_2^* is zero, that is in this case the worm will disappear. By Theorems 1-4, in almost all cases, the limits points are in a worm free equilibrium state. Any fixed point of the operator V is an equilibrium state and Theorems 1-4 give that system may have a continuum set of equilibrium states, the system stays in a neighborhood of one of the equilibrium states (stable fixed point), which depends on the initial state and on the parameters.

Thus the discrete-time dynamical systems generated by SIS, SIR, PSIDR models on basis QSOs also approximate the real conditions and computer simulations showed the close to the real data results. At the same time our simulation results allow practically repeat a real worm behavior. The discrete-time SIS and PSIDR models simulations showed the most close to the real data results as the continuous time models [6] and as the multiagent model which one introduced [5].

The results of analysis of discrete-time models on basis QSOs led to the conclusion about the appropriateness of such discrete models in computer networks.

Acknowledgements

We thank the referees for the helpful comments and suggestions that contributed to the improvement of this paper.

This work was done in the University of Latvia. The second author (UJ) thanks the University of Latvia, for the kind hospitality and for providing all facilities and the State Education Development Agency Republic of Latvia for providing financial support by a fellowship for research. The third author (AR) thanks the project of Institute of Mathematics and Computer Science "Dynamic equations on time scale".

References

[1] E. Akin and V. Losert. Evolutionary dynamics of zero-sum games. *Journal of Mathematical Biology*, **20**(3):231–258, 1984. https://doi.org/10.1007/BF00275987.

[2] L.J.S. Allen. Some discrete-time SI, SIR, and SIS epidemic models. *Mathematical BioSciences*, **124**(1):83–105, 1994. https://doi.org/10.1016/0025-5564(94)90025-6.

[3] V.M. Barashkov and N.A. Zadorina. Analysis of two-stage mathematical models of computer distribution viruses. In A.B. Yegorov, V.Ya. Hentov and at al.(Eds.),

Proc. of the 5th Intern. Conference Actual Problems Technical Sciences in Russia and abroad, Novosibirsk, Russia, 2018, pp. 96–102, 2018. (in Russian)

[4] J. Blath, U.U. Jamilov(Zhamilov) and M. Scheutzow. (G,μ)- quadratic stochastic operators. *Journal of Difference Equations and Applications*, **20**(8):1258–1267, 2014. https://doi.org/10.1080/10236198.2014.906591.

[5] D. Chumachenko, K. Chumachenko and S. Yakovlev. Intelligent simulation of network worm propagation using the code red as an example. *Telecommunications and Radio Engineering*, **78**(5):443–464, 2019. https://doi.org/10.1615/TelecomRadEng.v78.i5.60.

[6] D. Chumachenko and S. Yakovlev. Development of deterministic models of malicious software distribution in heterogeneous networks. In *2019 3rd International Conference on Advanced Information and Communications Technologies (AICT)*, pp. 439–442, 2019. https://doi.org/10.1109/AIACT.2019.8847903.

[7] Á.M. del Rey. A SIR e-epidemic model for computer worms based on cellular automata. In C. Bielza, A. Salmerón and et al.(Eds.), *Advances in Artificial Intelligence*, pp. 228–238, Berlin, Heidelberg, 2013. Proceedings of 15th Conference of the Spanish Association for Artificial Intelligence, CAEPIA 2013 Madrid, Spain, September 17-20, 2013.

[8] R.L. Devaney. *An introduction to chaotic dynamical systems*. Studies in Nonlinearity, Westview Press, Boulder, CO, 2003.

[9] N.N. Ganikhodjaev, R.N. Ganikhodjaev and U.U. Jamilov. Quadratic stochastic operators and zero-sum game dynamics. *Ergodic Theory and Dynamical Systems*, **35**(5):1443–1473, 2015. https://doi.org/10.1017/etds.2013.109.

[10] N.N. Ganikhodjaev, U.U. Jamilov and R.T. Mukhitdinov. On non - ergodic transformations on S^3. *Journal of Physics: Conference Series*, **435**(1):012005, 2013. https://doi.org/10.1088/1742-6596/435/1/012005.

[11] R.N. Ganikhodzhaev. Quadratic stochastic operators, Lyapunov functions and tournaments. *Russian Academy of Sciences. Sbornik Mathematics*, **76**(2):489–506, 1993. https://doi.org/10.1070/SM1993v076n02ABEH003423.

[12] R.N. Ganikhodzhaev. Map of fixed points and Lyapunov functions for one class of discrete dynamical systems. *Math. Notes*, **56**(5):1125–1131, 1994. https://doi.org/10.1007/BF02274660.

[13] R.N. Ganikhodzhaev, F.M. Mukhamedov and U.A. Rozikov. Quadratic stochastic operators and processes: results and open problems. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, **14**(2):279–335, 2011. https://doi.org/10.1142/S0219025711004365.

[14] U. Jamilov. On a family of strictly non-Volterra quadratic stochastic operators. *Journal of Physics: Conference Series*, **697**(1):012013, 2016. https://doi.org/10.1088/1742-6596/697/1/012013.

[15] U.U. Jamilov. Quadratic stochastic operators corresponding to graphs. *Lobachevskii Journal of Mathematics*, **34**(2):148–151, 2013. https://doi.org/10.1134/S1995080213020042.

[16] J. O. Kephart and S. R. White. Directed-graph epidemiological models of computer viruses. In *Proc. of the 1991 IEEE Computer Society Symposium on Research in Security and Privacy, Oakland, California*, pp. 343–359, 1991.

[17] H. Kesten. Quadratic transformations: A model for population growth. I. *Advances in Applied Probability*, **2**(1):1–82, 1970. https://doi.org/10.2307/3518344.

[18] I.V. Kotenko and V.V. Vorontsov. Analytical models of network worm propagation. In *SPIIRAS Proceedings*, volume 4, pp. 208–224, SPb., 2007. Nauka. https://doi.org/10.15622/sp.4.15. (in Russian)

[19] Y.I. Lyubich. *Mathematical structures in population genetics.* vol. 22 of Biomathematics, Springer-Verlag, Berlin, 1992.

[20] P. Van Mieghem. The n-intertwined SIS epidemic network model. *Computing*, **93**(2):147–169, 2011. https://doi.org/10.1007/s00607-011-0155-y.

[21] F. Mukhamedov and A.F. Embong. On stable of b-bistochastic quadratic stochastic operators and associated nonhomogenous Markov chains. *Linear and Multilinear Algebra*, **66**(1):1–21, 2018. https://doi.org/10.1080/03081087.2017.1281215.

[22] F. Mukhamedov and N. Ganikhodjaev. *Quantum Quadratic Operators and Processes.* vol. 2133, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2015. https://doi.org/10.1007/978-3-319-22837-2.

[23] F. Mukhamedov, M. Saburov and I. Qaralleh. On $\xi^{(s)}$-quadratic stochastic operators on two-dimensional simplex and their behavior. *Abst. Appl. Anal.*, **2013**:942038, 2013. https://doi.org/10.1155/2013/942038.

[24] U.A. Rozikov and S. K. Shoyimardonov. Ocean ecosystem discrete time dynamics generated by $l-$ Volterra operators. *International Journal of Biomathematics*, **12**(2):1950015, 2019. https://doi.org/10.1142/S1793524519500153.

[25] U.A. Rozikov and S.S. Xudayarov. Quadratic non-stochastic operators: examples of splitted chaos. *Annals of Functional Analysis*, **13**(1):17, 2022. https://doi.org/10.1007/s43034-021-00162-5.

[26] U.A. Rozikov and A. Zada. On dynamics of $l-$ Volterra quadratic stochastic operators. *International Journal of Biomathematics*, **3**(2):143–159, 2010. https://doi.org/10.1142/S1793524510000945.