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On the Regularity Criterion on One Velocity Component for the Micropolar Fluid Equations

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Abstract. In this paper, we establish a regularity criterion for micropolar fluid flows in terms of the one component of the velocity in critical Morrey-Campanato space. More precisely, we show that if

$$\int_0^T \|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{24}{9-10r}} d\tau < \infty, \quad \text{where } \ 0 < \ r < \frac{9}{10},$$

then the weak solution (u, w) is regular.

Keywords: micropolar fluid equations, weak solutions, regularity criterion, Morrey-Campanato spaces.

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1 Introduction

In this paper, we consider the following Cauchy problem of the three dimensional viscous incompressible micropolar fluid flows [10]:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = \nabla \times w, \\ \partial_t w - \Delta w - \nabla \text{div } w + (u \cdot \nabla)w + 2w = \nabla \times u, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \end{cases} \tag{1.1}$$

where u denotes the unknown velocity of the fluid, w is the angular velocity, and the scalar function π denotes the unknown pressure, while u_0 and w_0 are the prescribed initial data for the velocity and angular velocity with the property $\nabla \cdot u_0 = 0$.

The micropolar fluid model was first proposed by Eringen [10] which describes some physical phenomena that cannot be treated by the classical Navier-Stokes equations for the viscous incompressible fluid, for example, the motion of animal blood, liquid crystals and dilute aqueous polymer solutions etc. Galdi and Rionero [14] and Lukaszewicz [29] proved the existence of global-in-time weak solutions. Dong et al. [8] proved the existence of local-in-time strong solutions. Concerning the regularity criteria of solutions to equations (1.1) one may refer to the references [7, 8, 9, 12, 13, 30] and references therein.

In case $w = 0$, (1.1) reduces to the incompressible Navier-Stokes equations. The classical Prodi-Serrin conditions (see [22, 25, 26, 27, 28]) states that if

$$u \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \le q \le \infty, \tag{1.2}$$

then the solution is smooth. Note that the limiting case $u \in L^\infty(0,T; L^3(\mathbb{R}^3))$ was covered by Escauriaza et al. [16] only recently by using backward uniqueness method. As remarked in [32], the main progress of (1.2) is to assume only one velocity component, say u_3, to be regular, but not all of them. Neustupa and Penel [21] verified the regularity for suitable weak solutions if one velocity component is essentially bounded. In particular, it is proved in [20] if

$$u_3 \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} \le \frac{1}{2}, \quad 6 < q < \infty, \tag{1.3}$$

then the solution is regular on $(0,T)$. Later, this condition (1.3) was improved by Kukavica and Ziane [17] to be

$$u_3 \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \frac{5}{8}, \quad \frac{16}{5} < p \le \infty, \quad \frac{24}{5} < q \le \infty \tag{1.4}$$

Whence, Cao and Titi [1] extended (1.4) to be

$$u_3 \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} \le \frac{2}{3} + \frac{2}{3q}, \quad \frac{7}{2} < q \le \infty. \tag{1.5}$$

Later, this condition (1.5) was improved by Zhou and Pokorný [33] to

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p}+\frac{3}{q}=\frac{3}{4}+\frac{1}{2q}, \quad \frac{10}{3}<q\le\infty.$$

However, these conditions are not scaling invariant. Many efforts have been made on weakening the above criterion by imposing constraints only on partial components or directional derivatives of velocity field. See, for instance, [2, 5, 6, 33] and the references therein. In a recent work, Chae and Wolf [4] proved the following regularity criterion

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p}+\frac{3}{q}<1, \quad 3<q<\infty.$$

Motivated by the above cited papers on the regularity criteria on the Leray-Hopf weak solutions to the 3D incompressible Navier-Stokes equations, the purpose of the present paper is focused on the regularity criterion of weak solutions to the 3D micropolar fluids in terms of one component of the velocity in the framework of Morrey-Campanato spaces. Before stating the main result, let us first recall the definition of weak solutions the 3D micropolar fluids (1.1) (see [14, 29]).

Definition 1 [weak solutions]. Let $0<T<\infty$, $(u_0,w_0)\in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$ in $\mathbb{R}^3$. A measurable function (u,w) on $\mathbb{R}^3\times(0,T)$ is called a weak solution of system (1.1) on $(0,T)$ if (u,w) satisfies the following properties

(i) $(u,w)\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$;

(ii) (u,w) verifies (1.1) in the sense of distribution, i.e.,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3}(\partial_t\varphi+(u\cdot\nabla)\varphi)u\,dxdt+\int_0^T\int_{\mathbb{R}^3}\nabla\times w\varphi\,dxdt+\int_{\mathbb{R}^3}u_0\varphi(x,0)dx\\&\qquad=\int_0^T\int_{\mathbb{R}^3}\nabla u:\nabla\varphi\,dxdt;\\&\int_0^T\int_{\mathbb{R}^3}(\partial_t\phi+(u\cdot\nabla)\phi)w\,dxdt+\int_0^T\int_{\mathbb{R}^3}\nabla\times u\phi\,dxdt+\int_{\mathbb{R}^3}w_0\phi(x,0)dx\\&\quad=\int_0^T\int_{\mathbb{R}^3}\nabla w:\nabla\phi\,dxdt+\int_0^T\int_{\mathbb{R}^3}\text{div } w\text{ div }\phi\,dxdt+2\int_0^T\int_{\mathbb{R}^3}w\phi\,dxdt,\end{aligned}$$

for all $\varphi,\phi\in C_0^\infty(\mathbb{R}^3\times[0,T))$ with div $\varphi=0$. div $u=0$ in distribution sense, i.e.,

$$\int_0^T\int_{\mathbb{R}^3}u\cdot\nabla\varphi\,dxdt=0,$$

for all $\varphi\in C_0^\infty(\mathbb{R}^3\times[0,T))$.

(iii) (u,w) satisfies the energy inequality, i.e.,

$$\|u(\cdot,t)\|_{L^2}^2+\|w(\cdot,t)\|_{L^2}^2+2\int_\epsilon^t(\|\nabla u(\cdot,\tau)\|_{L^2}^2+\|\nabla w(\cdot,\tau)\|_{L^2}^2)d\tau$$
$$+2\int_\epsilon^t\|\mathrm{div}\ w(\cdot,\tau)\|_{L^2}^2\,d\tau+2\int_\epsilon^t\|w(\cdot,\tau)\|_{L^2}^2\,d\tau\le\|u(\epsilon)\|_{L^2}^2+\|w(\epsilon)\|_{L^2}^2\,, \qquad (1.6)$$

for $0\le\epsilon\le t\le T$.

Now, our main result read as follows:

Theorem 1. *Let $T>0$ be a given time and $(u_0,w_0)\in H^1(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$ in $\mathbb{R}^3$. Assume that (u,w) is a weak solution of system (1.1) on $\mathbb{R}^3\times(0,T)$. Suppose that u_3 satisfies*

$$u_3\in L^{\frac{24}{9-10r}}(0,T;\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)),\quad \text{where}\quad 0<\ r<9/10, \qquad (1.7)$$

then the solution (u,w) is regular on $\mathbb{R}^3\times(0,T)$.

Remark 1. Our main result concerns the regularity conditions which can guarantee the regularity of weak solutions involving only one velocity component u_3 in critical Morrey-Campanato space $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$. It was already known that if one component of the velocity is bounded in a suitable space, then the solution is smooth (see Penel and Pokorny [24], He [15], Zhou [31], Chae and Choe [3]). Notice that we add no conditions on the micro-rotational velocity w. If let $w=0$, (1.7) reduces to the well-known Serrin's regularity criterion for 3D incompressible Navier-Stokes equations. Therefore, Theorem 1 can be regarded as the Serrin type regularity criterion on 3D incompressible Navier-Stokes equations. A natural region for r in (1.7) should be $0\le r<1$ but we only can prove it for $0<r<\frac{9}{10}$ here. At present, we are not able to show that (1.7) still holds true for $\frac{9}{10}\le r<1$. The key reason is that the proof heavily relies on (2.2).

Remark 2. In our knowledge, this is the first regularity criterion result is concerned with weak solution to the 3D incompressible micropolar fluid flows in Morrey-Campanato space. The most difficulties that arising is to handle the nonlinear term $\int_{\mathbb{R}^3}(u\cdot\nabla)w\cdot\Delta w dx$. Fortunately, L^3-energy estimate for w helps us to overcome this problem.

Remark 3. Since $L^{\frac{3}{r}}(\mathbb{R}^3)\hookrightarrow\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$, it is clear that our result extend the Navier-Stokes equations' result to micropolar fluid equations.

As a corollary of Theorem 1, we get the following regularity criterion for the 3D Navier-Stokes equations in terms of one velocity component.

Corollary 1. Let $u_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$. Assume that u is a weak solution of system (1.1) on $\mathbb{R}^3\times[0,T)$. If

$$u_3\in L^{\frac{24}{9-10r}}(0,T;L^{\frac{3}{r},\infty}(\mathbb{R}^3)),\quad \text{for some } r \text{ with}\quad 0<\ r<9/10.$$

then, u is regular solution, where $L^{\frac{3}{r},\infty}$ denotes the weak-$L^{\frac{3}{r}}$ space.

1.1 Morrey-Campanato space

The classical Morrey spaces have been of great value through the years in studying the local behavior of solutions to second elliptic partial differential equations.

Definition 2. Let $0 \leq r < \frac{3}{2}$ be a real parameter. For $f \in L^2_{\text{loc}}(\mathbb{R}^3)$, define

$$\|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} = \sup_{Q\in\mathcal{Q}} |Q|^{\frac{r}{3}-\frac{1}{2}} \left(\int_Q |f(x)|^2 \, dx \right)^{\frac{1}{2}} = \sup_{Q\in\mathcal{Q}} |Q|^{\frac{r}{3}} \left(\frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{\frac{1}{2}},$$

where we have used the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^3$ with sides parallel to the coordinate axes and $|Q|$ to denote the volume of Q.

The Morrey-Camapanto space $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ is defined to be the subset of all L^2 locally integrable functions f on $\mathbb{R}^3$ for which $\|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}$ is finite. Furthermore, it is easy to check the following (see e.g. [12, 13]):

$$L^{\frac{3}{r}}(\mathbb{R}^3) = \dot{\mathcal{M}}_{\frac{3}{r},\frac{3}{r}}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3).$$

It can be seen from the special case $\dot{\mathcal{M}}_{\frac{3}{r},\frac{3}{r}}(\mathbb{R}^3) = L^{\frac{3}{r}}(\mathbb{R}^3)$ that Morrey – Camapanto space is the generalization of Lebesgue space. If we let $f(x) = |x|^{-\frac{3}{p}}$ then the cube $\mathcal{R} = (-\frac{t}{2}, \frac{t}{2})^3$, $t > 0$ attains its Morrey-Camapanto-norm. In fact, if $0 < r < \frac{3}{2}$, then

$$\sup_{Q\in\mathcal{Q}} |Q|^{\frac{r}{3}-\frac{1}{2}} \left(\int_Q \frac{1}{|x|^{2r}} dx \right)^{\frac{1}{2}} \leq C \, |\mathcal{R}|^{\frac{r}{3}-\frac{1}{2}} \left(\int_{\mathcal{R}} \frac{1}{|x|^{2r}} dx \right)^{\frac{1}{2}} = O\left((3-2r)^{-\frac{1}{2}} \right)$$

and f belongs to $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$. Because then f does not belong to $L^{\frac{3}{r}}(\mathbb{R}^3)$, we see that the Morrey-Camapanto space $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ is properly wider than the Lebesgue space $L^{\frac{3}{r}}(\mathbb{R}^3)$. The completeness of Morrey-Camapanto spaces follows easily by that of Lebesgue spaces. We also have the following comparison between the Lorentz spaces and the Morrey-Campanato spaces:

$$L^{\frac{3}{r}}\left(\mathbb{R}^3\right) \subset \ L^{\frac{3}{r},\infty}\left(\mathbb{R}^3\right) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}\left(\mathbb{R}^3\right).$$

The relation

$$L^{\frac{3}{r},\infty}\left(\mathbb{R}^3\right) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}\left(\mathbb{R}^3\right)$$

is shown in the following

$$\begin{aligned}
&\|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \leq \sup_{E} |E|^{\frac{r}{3}-\frac{1}{2}} \left(\int_E |f(y)|^2 \, dy \right)^{\frac{1}{2}} \qquad \left(f \in L^{\frac{3}{r},\infty}\left(\mathbb{R}^3\right) \right) \\
&= \left(\sup_{E} |E|^{\frac{2r}{3}-1} \int_E |f(y)|^2 \, dy \right)^{\frac{1}{2}} \cong \left(\sup_{R>0} R \left| \left\{ x \in \mathbb{R}^3 : |f(y)|^2 > R \right\} \right|^{\frac{2r}{3}} \right)^{\frac{1}{2}} \\
&= \sup_{R>0} R \left| \left\{ x \in \mathbb{R}^3 : |f(y)| > R \right\} \right|^{\frac{r}{3}} \cong \|f\|_{L^{\frac{3}{r},\infty}}.
\end{aligned}$$

We also need the predual of $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$.

Lemma 1. *[18] For $0 \leq r < \frac{3}{2}$, let the space $\mathcal{M}(\dot{B}^r_{2,1} \mapsto L^2)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Besov space $\dot{B}^r_{2,1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\mathcal{M}(\dot{B}^r_{2,1} \mapsto L^2)$ is given by the operator norm of pointwise multiplication:*

$$\|f\|_{\mathcal{M}(\dot{B}^r_{2,1}\mapsto L^2)} = \sup\{\|fg\|_{L^2} | \|g\|_{\dot{B}^r_{2,1}} \leq 1\}.$$

Then, f belongs to $\mathcal{M}(\dot{B}^r_{2,1} \mapsto L^2)$ if and only if f belongs to $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ (with equivalence of norms).

We recall the following three-dimensional Sobolev-Ladyzhenskaya inequality in the whole space $\mathbb{R}^3$ (see, for example [1]).

Lemma 2. *For $1 \leq q < \infty$, there exists a constant C such that for $f \in C_0^\infty(\mathbb{R}^3)$,*

$$\|f\|_{L^{3q}} \leq C \|\partial_3 f\|_{L^q}^{\frac{1}{3}} \|\nabla_h f\|_{L^2}^{\frac{2}{3}}, \tag{1.8}$$

where $\nabla_h = (\partial_1, \partial_2)$ is the horizontal gradient operator.

In order to prove our main result, we need the following result, which will play an important role in our discussion.

Lemma 3. *Let $(u_0, w_0) \in H^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0$ and (u, w) is a weak solution of to the system (1.1) on some interval $[0, T)$. Then*

$$\|w(\cdot,t)\|_{L^3}^3 + \frac{2}{3}\int_0^t \left\|\nabla |w|^{\frac{3}{2}}(\cdot,\tau)\right\|_{L^2}^2 d\tau \leq C(\|u_0\|_{L^2}, \|w_0\|_{L^2}, \|w_0\|_{L^3}, T),$$

for all $t \in [0, T)$.

Proof. In the calculations will be used the following equalities:

$$\int_{\mathbb{R}^3} (u\cdot\nabla)w \cdot w\,|w|\,dx = 0, \tag{1.9}$$

$$\int_{\mathbb{R}^3} (\nabla \times u)\cdot w dx = \int_{\mathbb{R}^3} u \cdot (\nabla \times w) dx, \tag{1.10}$$

$$\left|\nabla |w|^{\frac{3}{2}}\right| = \frac{3}{2}|w|^{\frac{1}{2}} |\nabla |w|| \quad \text{ a.e. in } \mathbb{R}^3, \tag{1.11}$$

$$\begin{aligned} -\int_{\mathbb{R}^3} (\Delta w)\cdot w\,|w|\,dx &= \int_{\mathbb{R}^3} \nabla(|w|\,w)\cdot\nabla w dx = \int_{\mathbb{R}^3} |w|\,|\nabla w|^2 dx \\ &+ \int_{\mathbb{R}^3} |w|\,|\nabla |w||^2 dx = \left\||w|^{\frac{1}{2}}|\nabla w|\right\|_{L^2}^2 + \frac{4}{9}\int_{\mathbb{R}^3}\left|\nabla |w|^{\frac{3}{2}}\right|^2 dx, \end{aligned} \tag{1.12}$$

$$\mathrm{div}(|w|\,w) = |w|\,\mathrm{div}\; w + w\cdot\nabla |w|. \tag{1.13}$$

Now, we multiply both sides of $(1.1)_2$ by $|w|\,w$ (as in the proof of Lemma 3.1, [23]) and then doing integration by parts in $\mathbb{R}^3$, taking into account (1.9), (1.10), (1.12), (1.13), it follows that

$$\frac{1}{3}\frac{d}{dt}\left\|w(\cdot,t)\right\|_{L^3}^3+\left\||w|^{\frac{1}{2}}|\nabla w|\right\|_{L^2}^2+\frac{4}{9}\left\|\nabla\left|w\right|^{\frac{3}{2}}\right\|_{L^2}^2+\left\||w|^{\frac{1}{2}}|\mathrm{div}\ w|\right\|_{L^2}^2+2\left\|w\right\|_{L^3}^3$$
$$\leq \int_{\mathbb{R}^3} u\cdot[\nabla\times(|w|\,w)]dx+\int_{\mathbb{R}^3}\mathrm{div}\ w\cdot(w\nabla\left|w\right|)dx=I+J, \tag{1.14}$$

Now, taking into account (1.11) to the last term in the right-hand side of (1.14), we get

$$|J|=\left|\int_{\mathbb{R}^3}(\mathrm{div}w)\cdot(w\nabla\left|w\right|)dx\right|\leq\int_{\mathbb{R}^3}\left|\mathrm{div}w\right|\left|w\right|\left|\nabla\left|w\right|\right|dx\leq\left\||w|^{\frac{1}{2}}\left|\left|\mathrm{div}w\right|\right|\right\|_{L^2}$$
$$\times\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}\leq\frac{2}{3}\left\||w|^{\frac{1}{2}}\left|\left|\mathrm{div}w\right|\right|\right\|_{L^2}\left\|\left|\nabla\left|w\right|^{\frac{3}{2}}\right|\right\|_{L^2}. \tag{1.15}$$

Then, by using the Young's inequality, from (1.15) we have

$$|J|\leq\frac{1}{3}\left\||w|^{\frac{1}{2}}\left|\left|\mathrm{div}w\right|\right|\right\|_{L^2}^2+\frac{1}{3}\left\|\left|\nabla\left|w\right|^{\frac{3}{2}}\right|\right\|_{L^2}^2\leq\left\||w|^{\frac{1}{2}}\left|\left|\mathrm{div}w\right|\right|\right\|_{L^2}^2+\frac{1}{3}\left\|\left|\nabla\left|w\right|^{\frac{3}{2}}\right|\right\|_{L^2}^2.$$

By using the facts that $|\nabla\left|w\right||\leq|\nabla w|$, $|\nabla\times w|\leq|\nabla w|$ and the following identity

$$\int_{\mathbb{R}^3}(\nabla\times u)\cdot|w|\,wdx=\int_{\mathbb{R}^3}u\cdot[\nabla\times(|w|\,w)]dx$$
$$=\int_{\mathbb{R}^3}u\cdot(|w|\,\nabla\times w)dx+\int_{\mathbb{R}^3}u\cdot(\nabla\left|w\right|\times w)dx$$
$$\leq\int_{\mathbb{R}^3}|u|\,|w|\,|\nabla w|\,dx+\int_{\mathbb{R}^3}|u|\,|\nabla\left|w\right|\times w|\,dx$$
$$\leq\int_{\mathbb{R}^3}|u|\,|w|\,|\nabla w|\,dx+\int_{\mathbb{R}^3}|u|\,|\nabla w|\,|w|\,dx\leq 2\int_{\mathbb{R}^3}|u|\,|w|\,|\nabla w|\,dx,$$

one can estimate I by using the Hölder inequality and the Young inequality

$$|I|\leq\int_{\mathbb{R}^3}|u|\,|w|\,|\nabla w|\,dx\leq\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}\left\||w|^{\frac{1}{2}}\left|u\right|\right\|_{L^2}\leq\frac{1}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2$$
$$+C\left\||w|^{\frac{1}{2}}\left|u\right|\right\|_{L^2}^2\leq\frac{1}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2+C\left\|w\right\|_{L^3}\left\|u\right\|_{L^3}^2$$
$$\leq\frac{1}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2+C\left\|w\right\|_{L^3}\left\|u\right\|_{L^2}\left\|\nabla u\right\|_{L^2}$$
$$\leq\frac{1}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2+C\left\|w\right\|_{L^3}\left\|u_0\right\|_{L^2}\left\|\nabla u\right\|_{L^2}$$
$$\leq\frac{1}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2+2\left\|w\right\|_{L^3}^3+C\left\|u_0\right\|_{L^2}^{\frac{3}{2}}\left\|\nabla u\right\|_{L^2}^{\frac{3}{2}}$$
$$\leq\frac{1}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2+2\left\|w\right\|_{L^3}^3+C(\left\|\nabla u\right\|_{L^2}^2+\left\|u_0\right\|_{L^2}^6).$$

Consequently, we have

$$\frac{d}{dt}\left\|w\right\|_{L^3}^3+\frac{3}{2}\left\||w|^{\frac{1}{2}}\left|\nabla w\right|\right\|_{L^2}^2+\frac{1}{3}\left\|\nabla\left|w\right|^{\frac{3}{2}}\right\|_{L^2}^2\leq C(\left\|\nabla u\right\|_{L^2}^2+\left\|u_0\right\|_{L^2}^6), \tag{1.16}$$

and integrating (1.16) over $(0,t)$, we obtain

$$\|w(\cdot,t)\|_{L^3}^3+\frac{2}{3}\int_0^t\left\|\nabla|w|^{\frac{3}{2}}(\cdot,\tau)\right\|_{L^2}^2d\tau\le\|w_0\|_{L^3}^3$$
$$+C\int_0^t(\|u_0\|_{L^2}^6+\|\nabla u(\cdot,\tau)\|_{L^2}^2)d\tau\le C(\|u_0\|_{L^2},\|w_0\|_{L^3},\|w_0\|_{L^2},T),$$

where we have used the energy inequality (1.6) in the last inequality. This completes the proof of Lemma 3. □

2 Proof of the Theorem 1

Since the initial data $(u_0,w_0)\in H^1(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)$ and $\nabla\cdot u_0=0$, there exists a unique local strong solution (u,w) of the 3D micropolar equations on $(0,T)$ (refer to Galdi and Rionero [14] or Lukaszewicz [29]), thus the proof of Theorem 1 is reduced to establish regular estimates uniformly on $(0,T)$, and then the local strong solution (u,w) can be continuously extended to the time $t=T$ by a standard continuation process (see, e.g., [8] or [11]). Therefore, in what follows, we may as well assume that the solution (u,w) is sufficiently smooth on $(0,T)$.

Proof. Applying ∇_h to $(1.1)_1$ and take the L^2 inner product of the resulting equations with $\nabla_h u$, with help of integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h u\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla_h(u\cdot\nabla)u\cdot\nabla_h udx$$
$$+\int_{\mathbb{R}^3}\nabla_h(\nabla\times w)\cdot\nabla_h udx=I+J. \tag{2.1}$$

At last, Y. Zhou and M. Pokorný [33] (see also [2]) proved that

$$\left|-\int_{\mathbb{R}^3}\nabla_h(u\cdot\nabla)u\cdot\nabla_h udx\right|\le C\int_{\mathbb{R}^3}|u_3|\,|\nabla u|\,|\nabla\nabla_h u|\,dx.$$

With the use of the Lemma 1 and the Young inequality, we derive the estimate of I as follows :

$$\left|-\int_{\mathbb{R}^3}\nabla_h(u\cdot\nabla)u\cdot\nabla_h udx\right|\le C\|u_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\|\nabla u\|_{\dot{B}_{2,1}^r}\|\nabla\nabla_h u\|_{L^2}$$
$$\le C\|u_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\|\nabla u\|_{L^2}^{1-r}\|\nabla u\|_{L^6}^{r}\|\nabla\nabla_h u\|_{L^2}\le C\|u_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\|\nabla u\|_{L^2}^{1-r}$$
$$\times\|\Delta u\|_{L^2}^{\frac{r}{3}}\|\nabla\nabla_h u\|_{L^2}^{1+\frac{2r}{3}}\le\|u_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{6}{3-2r}}\|\nabla u\|_{L^2}^{2-\frac{2r}{3-2r}}\|\Delta u\|_{L^2}^{\frac{2r}{3-2r}}+\frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2,$$

where we have used the following interpolation inequality in Besov spaces [19]:

$$\|f\|_{\dot{B}_{2,1}^r}\le C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r\quad\text{with }\ 0<r<1. \tag{2.2}$$

Thanks to the Hölder and Young inequality, one deduces

$$J=\int_{\mathbb{R}^3}\sum_{k=1}^{2}\partial_k(\nabla\times w)\cdot\partial_k u dx=-\int_{\mathbb{R}^3}\sum_{k=1}^{2}(\nabla\times w)\cdot\partial_k\partial_k u dx$$
$$\leq\|\nabla\times w\|_{L^2}\|\nabla_h\nabla u\|_{L^2}\leq C\|\nabla w\|_{L^2}^2+\frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2.$$

Inserting all the estimates into (2.1), we get

$$\frac{d}{dt}\|\nabla_h u(t)\|_{L^2}^2+\|\nabla\nabla_h u\|_{L^2}^2\leq C\|u_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{6}{3-2r}}\|\nabla u\|_{L^2}^{2-\frac{2r}{3-2r}}\|\Delta u\|_{L^2}^{\frac{2r}{3-2r}}+C\|\nabla w\|_{L^2}^2.$$

Integrating over $[0,t)$, one can verify

$$\sup_{\tau\in[0,t]}\|\nabla_h u(\tau)\|_{L^2}^2+\int_0^t\|\nabla\nabla_h u(\tau)\|_{L^2}^2 d\tau$$
$$\leq\ \|\nabla_h u_0\|_{L^2}^2+C\int_0^t\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{6}{3-2r}}\|\nabla u(\tau)\|_{L^2}^{2-\frac{2r}{3-2r}}\|\Delta u(\tau)\|_{L^2}^{\frac{2r}{3-2r}}d\tau. \quad (2.3)$$

Now, we will establish the bounds of H^1-norm of the velocity and micro-rotational velocity. In order to do it, taking the L^2 inner product of the equations (1.1) with $-\Delta u$ and $-\Delta w$ respectively, and by integrating by parts and using the incompressibility condition, we have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta w\|_{L^2}^2+\|\nabla\mathrm{div}w\|_{L^2}^2+2\|\nabla w\|_{L^2}^2$$
$$=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx+\int_{\mathbb{R}^3}(u\cdot\nabla)w\cdot\Delta w dx$$
$$-\int_{\mathbb{R}^3}(\nabla\times u)\cdot\Delta w dx-\int_{\mathbb{R}^3}(\nabla\times w)\cdot\Delta u dx=-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j dx$$
$$-\int_{\mathbb{R}^3}\nabla u\cdot\nabla w\cdot\nabla w dx+2\int_{\mathbb{R}^3}\nabla(\nabla\times u)\cdot\nabla w dx. \quad (2.4)$$

Integration by parts implies that

$$\int_{\mathbb{R}^3}(u\cdot\nabla)w\cdot\Delta w dx = -\int_{\mathbb{R}^3}\nabla u\cdot\nabla w\cdot\nabla w dx$$
$$=\int_{\mathbb{R}^3}\nabla(\nabla u)\cdot\nabla w\cdot w dx+\int_{\mathbb{R}^3}\nabla u\,\nabla(\nabla w)\cdot w dx. \quad (2.5)$$

We now estimate the above terms one by one. To bound $\int_{\mathbb{R}^3}\nabla(\nabla u)\cdot\nabla w\cdot w dx$, we first integrate by parts and then apply Hölder's inequality to obtain

$$\left|\int_{\mathbb{R}^3}\nabla(\nabla u)\cdot\nabla w\cdot w dx\right| \leq \int_{\mathbb{R}^3}|\Delta u|\,|\nabla w|\,|w|\,dx$$
$$\leq \|\Delta u\|_{L^2}\|w\|_{L^s}\|\nabla w\|_{L^{\frac{2s}{s-2}}}. \quad (2.6)$$

It follows from the Gagliardo-Nirenberg inequality ($3 \leq s \leq 9$) that

$$\|w\|_{L^s} \leq C\|w\|_{L^3}^{\frac{9-s}{2s}} \|w\|_{L^9}^{\frac{3s-9}{2s}}, \qquad \|\nabla w\|_{L^{\frac{2s}{s-2}}} \leq C\|\nabla w\|_{L^2}^{1-\frac{3}{s}} \|\Delta w\|_{L^2}^{\frac{3}{s}},$$

and from Lemma 3 that

$$\|w\|_{L^3}^3 \leq C(\|u_0\|_{L^2}, \|w_0\|_{L^2}, T), \quad \text{for all } 0 \leq t < T.$$

Substituting these two estimates into (2.6) and then using Young's inequality, we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \nabla(\nabla u)\cdot\nabla w \cdot w dx\right| &\leq C\|\Delta u\|_{L^2} \left\||w|^{\frac{3}{2}}\right\|_{L^6}^{1-\frac{3}{s}} \|\nabla w\|_{L^2}^{1-\frac{3}{s}} \|\Delta w\|_{L^2}^{\frac{3}{s}} \\ &\leq C\|\Delta u\|_{L^2} \left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^{1-\frac{3}{s}} \|\nabla w\|_{L^2}^{1-\frac{3}{s}} \|\Delta w\|_{L^2}^{\frac{3}{s}} \\ &\leq C\left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^2 \|\nabla w\|_{L^2}^2 + \frac{1}{4}\|\Delta w\|_{L^2}^2 + \frac{1}{6}\|\Delta u\|_{L^2}^2.\end{aligned}$$

Applying similar procedure, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \nabla u\cdot\nabla(\nabla w)\cdot w dx\right| &\leq \int_{\mathbb{R}^3} |\nabla u|\,|\Delta w|\,|w|\,dx \leq \|\Delta w\|_{L^2}\|w\|_{L^s}\|\nabla u\|_{L^{\frac{2s}{s-2}}} \\ &\leq C\left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Delta w\|_{L^2}^2 + \frac{1}{6}\|\Delta u\|_{L^2}^2.\end{aligned}$$

Substituting these two estimates into (2.5) and then using Young's inequality, we see that

$$\begin{aligned}\left|-\int_{\mathbb{R}^3} \nabla u\cdot\nabla w\cdot\nabla w dx\right| &\leq C\left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\ &+ \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{3}\|\Delta u\|_{L^2}^2. \end{aligned} \tag{2.7}$$

For the term $2\int_{\mathbb{R}^3} \nabla(\nabla\times u)\cdot\nabla w dx$, clearly one has

$$2\left|\int_{\mathbb{R}^3} \nabla(\nabla\times u)\cdot\nabla w dx\right| \leq 2\|\nabla w\|_{L^2}^2 + \frac{1}{6}\|\Delta u\|_{L^2}^2. \tag{2.8}$$

Inserting (2.7)–(2.8) into (2.4), using the Hölder inequality, the interpolation inequality, and (1.8), it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) + \frac{1}{2}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\|\Delta w\|_{L^2}^2 + \|\nabla \mathrm{div} w\|_{L^2}^2 \\ &\leq C\int_{\mathbb{R}^3} |\nabla_h u|\,|\nabla u|^2 dx + C\left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\ &\leq C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^4}^2 + C\left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \leq C\|\nabla_h u\|_{L^2} \\ &\times \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}} + C\left\|\nabla |w|^{\frac{3}{2}}\right\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2).\end{aligned}$$

Integrating on $[0,t]$, it leads to the following estimate

$$\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\int_0^t(\|\Delta u(\tau)\|_{L^2}^2+\|\Delta w(\tau)\|_{L^2}^2+2\,\|\nabla \mathrm{div} w(\tau)\|_{L^2}^2)d\tau$$
$$\le \|\nabla u_0\|_{L^2}^2+\|\nabla w_0\|_{L^2}^2+C\int_0^t\|\nabla_h u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}^{\frac12}\|\nabla\nabla_h u(\tau)\|_{L^2}$$
$$\times\|\Delta u(\tau)\|_{L^2}^{\frac12}d\tau+C\int_0^t\left\|\nabla|w|^{\frac32}(\tau)\right\|_{L^2}^2(\|\nabla u(\tau)\|_{L^2}^2+\|\nabla w(\tau)\|_{L^2}^2)d\tau. \tag{2.9}$$

Set

$$\mathcal{J}(t)=\int_0^t\|\nabla_h u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}^{\frac12}\|\nabla\nabla_h u(\tau)\|_{L^2}\|\Delta u(\tau)\|_{L^2}^{\frac12}d\tau.$$

Thanks to (u,w) is a weak solution of the Equations (1.1), then

$$\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau\le\|u_0\|_{L^2}^2,$$

which together with (2.3) yields that

$$\mathcal{J}(t)\le C\left(\sup_{\tau\in[0,t]}\|\nabla_h u(\tau)\|_{L^2}\right)\left(\int_0^t\|\nabla\nabla_h u(\tau)\|_{L^2}^2d\tau\right)^{\frac12}\left(\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau\right)^{\frac14}$$
$$\times\left(\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau\right)^{\frac14}\le C\left(\sup_{\tau\in[0,t]}\|\nabla_h u(\tau)\|_{L^2}^2+\int_0^t\|\nabla\nabla_h u(\tau)\|_{L^2}^2d\tau\right)$$
$$\times\left(\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau\right)^{\frac14}\le C\Big(\|\nabla_h u_0\|_{L^2}^2+\int_0^t\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac3r}}^{\frac{6}{3-2r}}\|\nabla u(\tau)\|_{L^2}^{2-\frac{2r}{3-2r}}$$
$$\times\|\Delta u(\tau)\|_{L^2}^{\frac{2r}{3-2r}}d\tau\Big)\left(\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau\right)^{\frac14}$$
$$\le C\|\nabla_h u_0\|_{L^2}^{\frac83}+C\left(\int_0^t\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac3r}}^{\frac{6}{3-2r}}\|\nabla u(\tau)\|_{L^2}^{2-\frac{2r}{3-2r}}\|\Delta u(\tau)\|_{L^2}^{\frac{2r}{3-2r}}d\tau\right)^{\frac43}$$
$$+\frac14\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau\le C\|\nabla_h u_0\|_{L^2}^{\frac83}$$
$$+C\left(\int_0^t\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac3r}}^{\frac{2}{1-r}}\|\nabla u(\tau)\|_{L^2}^2d\tau\right)^{\frac{12(1-r)}{9-10r}}+\frac12\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau$$
$$\le C\|\nabla_h u_0\|_{L^2}^{\frac83}+C\left(\int_0^t\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac3r}}^{\frac{24}{9-10r}}\|\nabla u(\tau)\|_{L^2}^2d\tau\right)\left(\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau\right)^{\frac{3-2r}{9-10r}}$$
$$+\frac12\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau\le C\|\nabla_h u_0\|_{L^2}^{\frac83}$$
$$+C\int_0^t\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac3r}}^{\frac{24}{9-10r}}\|\nabla u(\tau)\|_{L^2}^2d\tau+\frac12\int_0^t\|\Delta u(\tau)\|_{L^2}^2d\tau. \tag{2.10}$$

Inserting (2.10) into (2.9) we get

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \int_0^t (\|\Delta u(\tau)\|_{L^2}^2 + \|\Delta w(\tau)\|_{L^2}^2) d\tau \\
&\quad \le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} \\
&\quad + C\int_0^t (\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{24}{9-10r}} + \left\|\nabla |w|^{\frac{3}{2}}(\tau)\right\|_{L^2}^2)(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2) d\tau.
\end{aligned}$$

By using Gronwall's inequality, the energy inequality we conclude that

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2) + \int_0^T (\|\Delta u(\tau)\|_{L^2}^2 + \|\Delta w(\tau)\|_{L^2}^2) d\tau \\
&\le \left[\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}}\right] e^{C\int_0^T (\|u_3(\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{24}{9-10r}} + \left\|\nabla |w|^{\frac{3}{2}}(\tau)\right\|_{L^2}^2) d\tau} < \infty.
\end{aligned}$$

We get by using the assumption (1.7) that $(u, w) \in L^\infty(0, T; H^1(\mathbb{R}^3))$. This completes the proof of Theorem 1. □

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