Mathematical Modelling and Analysis
Volume 27, Issue 2, 215–231, 2022
https://doi.org/10.3846/mma.2022.14194

An Effective and Simple Scheme for Solving Nonlinear Fredholm Integral Equations

Ahmad Shahsavaran and Forough Fotros

Young Researchers and Elite Club, Borujerd Branch, Islamic Azad University
Takhti Street, Borujerd, Iran
E-mail(*corresp.*): a.shahsavaran@iaub.ac.ir
E-mail: forough.fotros@yahoo.com

Received January 8, 2021; revised February 14, 2022; accepted February 14, 2022

Abstract. In this paper, a simple scheme is constructed for finding approximate solution of the nonlinear Fredholm integral equation of the second kind. To this end, the Lagrange interpolation polynomials together with the Gauss-Legendre quadrature rule are used to transform the source problem to a system of nonlinear algebraic equations. Afterwards, the resulting system can be solved by the Newton method. The basic idea is to choose the Lagrange interpolation points to be the same as the points for the Gauss-Legendre integration. This facilitates the evaluation of the integral part of the equation. We prove that the approximate solution converges uniformly to the exact solution. Also, stability of the approximate solution is investigated. The advantages of the method are simplicity, fastness and accuracy which enhance its applicability in practical situations. Finally, we provide some test examples.

Keywords: Fredholm integral equation, Lagrange polynomials, Gauss-Legendre integration, interpolation, convergence and stability.

AMS Subject Classification: 45B05; 45G10; 65R20.

1 Introduction

This paper concerns the construction of efficient and stable numerical method for solving the nonlinear Fredholm integral equation of the form

$$u(x) = f(x) + \int_{-1}^{1} K(x,t)\phi(u(t))dt, \quad x \in [-1,1], \tag{1.1}$$

where $f \in C[-1,1]$ and $K \in C([-1,1]^2)$ are known functions, ϕ is a given nonlinear function defined on $[-1,1]$ and u is unknown to be determined.

Various phenomena in different fields of science and engineering can be formulated by integral equations [37]. Fredholm integral equations appear in numerous applications such as thermal radiation [20], fluid mechanics [10], simulation of tumor growth [9], transport theory [34], image restoration [24], potential theory and elasticity [23], etc.

Integral equations are usually difficult to solve analytically. Therefore, this requires that its solution to be obtained approximately. In recent years, numerical methods for solving different types of the Fredholm integral equations have been extensively studied in many papers such as Legendre multi-Galerkin methods to solve the Fredholm integral equation with weakly singular kernel [30], iterative procedure for the nonlinear fuzzy Fredholm integral equation [15], Chebyshev collocation method for a class of the Fredholm integral equations with highly oscillatory kernels [18], operator transformation for the Fredholm integral equation with Cauchy type kernel [7], spectral collocation method for the Fredholm integral equation on the half-line [31], Nyström method for the Fredholm integral equation on unbounded domain [17], Chebyshev approximation for the nonlinear Fredholm-Volterra integral equation [16], Legendre polynomials for the nonlinear Fredholm-Hammerstein integral equation [28], Legendre wavelets for the system of linear and nonlinear Fredholm integral equations [13], Legendre spectral collocation method for the nonlinear Fredholm integral equation in multidimensions [38], Newton-Steffensen iterative scheme for the nonlinear Fredholm integral equation with non-differentiable Nemystkii operator [19], Nyström method for the nonlinear integro-differential equation of Fredholm type [8], etc. For a review on different numerical techniques to solve such problems, see for instance [2,5,33]. Existence and uniqueness of such equations are discussed in [11].

Interpolation techniques and quadrature methods have been widely used to solve the nonlinear Fredholm integral equations. Katani in [22] used the Romberg quadrature rule to find the numerical solution of the Fredholm integral equation of Urysohn type. Molabahrami applied the Lagrange interpolation method for the system of Urysohn type integral equations [27]. Derakhshan and Zarebnia [12] constructed quadrature rules of arbitrary low and high orders based on quadratic spline quasi-interpolant to approximate the solution of (1.1). Assari in [3] used the discrete Galerkin method based on the moving least squares approach to approximate the solution of one and two dimensional integral equations of the form (1.1). In [4], a method based on the discrete collocation method together with the radial basis functions was also presented by this author to find the numerical solution of (1.1). In both methods, the required integrals estimated using the Gauss-Legendre quadrature rule. For further references in that respect, see e.g., [1, 21]. In this study, we present an efficient and stable numerical method based on the Lagrange interpolation together with the Gauss-Legendre quadrature rule to find the numerical solution of (1.1). The main advantage of the presented method is that the source problem is reduced to a system of nonlinear algebraic equations. The resulting system can be solved by the Newton method.

The remainder of this paper is organized as follows. In Section 2, we introduce the Gauss-Legendre quadrature formula and also the Lagrange interpo-

lation. In Section 3, we explain the numerical method to solve equation (1.1) using what was introduced in Section 2. In Section 4, convergence and stability of approximate solution are considered. Some test examples are provided in Section 5 to demonstrate the applicability of the method. Finally, a conclusion is given in Section 6.

2 Gauss-Legendre quadrature formula and Lagrange interpolation

An Gauss-Legendre quadrature formula of order $k+1$ is given by

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{k} \omega_i f(x_i), \tag{2.1}$$

for some set of nodes $\{x_i\}_{i=0}^{k}$ and weights $\{\omega_i\}_{i=0}^{k}$. The Gauss-Legendre quadrature rule is defined by the unique choice for the nodes and weights so that (2.1) is exact when f is any polynomial of degree at most $2k+1$. No explicit formula is known for the points x_i and so they are calculated numerically. The nodes for quadrature of order $k+1$ are given by the roots of the Legendre polynomial $p_{k+1}(x)$ which occur symmetrically about 0. The points x_i are available in most standard mathematical tables and computer codes. Also, the weights ω_i are given by

$$\omega_i = \frac{2(1-x_i^2)}{(k+1)^2 \left(p_{k+1}(x_i)\right)^2}, \quad i = 0, 1, ..., k.$$

More information and details about the Gaussian quadrature formula of order k and calculated values of corresponding nodes and weights would be found in [36].

It should be noted that when using the extensively tabulated roots of the Legendre polynomials and their corresponding weights to solve an equation whose integration interval is $[a, b]$, we need to first apply the following change of variables

$$x = \frac{b-a}{2}\overline{x} + \frac{b+a}{2},$$

which gives

$$\int_a^b f(x)dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2}\bar{x} + \frac{b+a}{2}\right) d\overline{x} \approx \frac{b-a}{2} \sum_{i=0}^{k} \omega_i f\left(\frac{b-a}{2}\bar{x}_i + \frac{b+a}{2}\right).$$

Now consider a set of $k+1$ distinct data points $(z_0, f_0), (z_1, f_1), ..., (z_k, f_k)$, then the Lagrange interpolation polynomial is given by

$$f(x) = \sum_{i=0}^{k} f_i \ell_{i,k}(x),$$

where $\ell_{i,k}(x)$, $i = 0, 1, ..., k$ are the Lagrange fundamental polynomials defined as

$$\ell_{i,k}(x) = \prod_{j=0, j\neq i}^{k} \left(\frac{x - z_j}{z_i - z_j}\right).$$

Also $\ell_{i,k}(z_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta, defined as

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

which simply implies that $f_i = f(z_i)$.

3 Method of solution

Consider the nonlinear Fredholm integral equation of the second kind

$$u(x) = f(x) + \int_{-1}^{1} K(x,t)\phi(u(t))dt, \quad |x| \leq 1. \tag{3.1}$$

This can be written as

$$u(x) = f(x) + \int_{-1}^{1} K(x,t)\Phi(t)dt, \tag{3.2}$$

where $\Phi(t) = \phi(u(t))$. By considering $u_k(x)$ as an approximation for $u(x)$ we can turn Equation (3.2) into the following equation

$$u_k(x) = f(x) + \int_{-1}^{1} K(x,t)\Phi_k(t)dt, \tag{3.3}$$

where $\Phi_k(t) = \phi(u_k(t))$ which immediately implies

$$\Phi_k(t) = \phi\left(f(t) + \int_{-1}^{1} K(t,x)\Phi_k(x)dx\right). \tag{3.4}$$

In fact, the first step to approximate the solution $u(x)$ of (3.1) is acquiring $\Phi_k(x)$. To this end, we consider the Lagrange interpolation of the function $\Phi_k(x)$ as

$$\Phi_k(x) = \sum_{i=0}^{k} \alpha_i \ell_{i,k}(x), \tag{3.5}$$

where $\alpha_i = \Phi_k(z_i)$. Here, we take the Lagrange interpolation points z_i to be the same as the points x_i for the Gauss quadrature formula. Now, using (3.5) and the quadrature formula (2.1), we may approximate the integral part of (3.4) as follows

$$\begin{aligned} &\int_{-1}^{1} K(t,x)\Phi_k(x)dx = \sum_{i=0}^{k} \alpha_i \int_{-1}^{1} K(t,x)\ell_{i,k}(x)dx \\ &= \sum_{i=0}^{k} \alpha_i \left(\sum_{j=0}^{k} \omega_j K(t,x_j)\ell_{i,k}(x_j)\right) = \sum_{i=0}^{k} \alpha_i \left(\sum_{j=0}^{k} \omega_j K(t,x_j)\delta_{ij}\right) \\ &= \sum_{i=0}^{k} \alpha_i \omega_i K(t,x_i). \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.4) gives

$$\Phi_k(t) = \phi\left(f(t) + \sum_{i=0}^{k} \alpha_i \omega_i K(t, x_i)\right). \tag{3.7}$$

Evaluating (3.7) at the points $t = x_p$, $p = 0, 1, ..., k$ (the points for the Gauss quadrature formula) implies

$$\Phi_k(x_p) = \phi\left(f(x_p) + \sum_{i=0}^{k} \alpha_i \omega_i K(x_p, x_i)\right). \tag{3.8}$$

Since $\Phi_k(x_p) = \alpha_p$, (3.8) is rewritten as

$$\alpha_p = \phi\left(f(x_p) + \sum_{i=0}^{k} \alpha_i \omega_i K(x_p, x_i)\right), \quad p = 0, 1, ..., k, \tag{3.9}$$

which is a nonlinear system of algebraic equations that can be solved by numerical methods such as Newton's method. By solving (3.9), the values α_p, $p = 0, 1, ..., k$, will be known. Now combining (3.3) and (3.6), we get

$$u_k(x) = f(x) + \sum_{i=0}^{k} \alpha_i \omega_i K(x, t_i), \tag{3.10}$$

where $t_i = x_i$, $i = 0, 1, ..., k$.

4 Convergence and stability

In this section, we give an estimation of the error bound for the proposed method of Section 3, which enables us to control the estimated errors. First, we provide the interpolation error formula which is given in the following theorem.

Theorem 1. *(Error formula for polynomial interpolation [35])*
Suppose $f \in C^{k+1}[-1, 1]$ and let p_k denote the polynomial that interpolates $\{(x_i, f(x_i))\}_{i=0}^{k}$ with $x_i \in [-1, 1]$ for $i = 0, 1, ..., k$. Then for every $x \in [-1, 1]$ there exists $\zeta_x \in [-1, 1]$ such that

$$f(x) - p_k(x) = \frac{\pi(x)}{(k+1)!} f^{(k+1)}(\zeta_x),$$

where $\pi(x) = \Pi_{i=0}^{k}(x - x_i)$.

As mentioned in Section 3, the Lagrange interpolation points $z_i, i=0, 1, ..., k$, are chosen to be respectively the same as the nodes x_i, $i = 0, 1, ..., k$, for the Gauss-Legendre quadrature formula, so we can provide the following theorem.

Theorem 2. *If the Lagrange interpolation polynomial $p_k(x)$ interpolates $f(x)$ at the Gauss-Legendre quadrature nodes x_i, then we have $|\pi(x)| < 1$ for $x \in [-1, 1]$.*

Proof. Let us consider two cases:

Case 1.

Since the nodes x_i are located symmetrically in $[-1, 1]$, then for odd k (the number of nodes is even), the nodes are x_0, x_1, ..., $x_{\frac{k-1}{2}}$, $x_{\frac{k+1}{2}}$, ..., x_{k-1}, x_k, which satisfy

$$x_k = -x_0,\ x_{k-1} = -x_1,\ ...,\ x_{\frac{k+1}{2}} = -x_{\frac{k-1}{2}}.$$

So,

$$\begin{aligned}\pi(x) &= \prod_{i=1}^{k}(x - x_i)\\ &= (x - x_0)(x - x_1)...(x - x_{\frac{k-1}{2}})(x - x_{\frac{k+1}{2}})...(x - x_{k-1})(x - x_k)\\ &= (x - x_0)(x - x_1)...(x - x_{\frac{k-1}{2}})(x + x_{\frac{k-1}{2}})...(x + x_1)(x + x_0)\\ &= (x^2 - x_0^2)(x^2 - x_1^2)...(x^2 - x_{\frac{k-1}{2}}^2), \quad |x| \le 1.\end{aligned}$$

Hence,

$$|\pi(x)| \le \left|(1 - x_0^2)(1 - x_1^2)...(1 - x_{\frac{k-1}{2}}^2)\right|. \tag{4.1}$$

As the nodes x_i satisfy $0 < x_i^2 < 1$ (no node equals 0 for odd k), we have

$$0 < 1 - x_i^2 < 1 \quad \text{for} \quad i = 0, 1, ..., (k-1)/2.$$

Therefore, (4.1) implies

$$|\pi(x)| < 1.$$

Case 2.

For even k (the number of nodes is odd), the nodes are x_0, x_1, ..., $x_{\frac{k}{2}-1}$, $x_{\frac{k}{2}}$, $x_{\frac{k}{2}+1}$, ..., x_{k-1}, x_k. For even k, $x_{\frac{k}{2}} = 0$ and for other nodes we have

$$x_k = -x_0,\ x_{k-1} = -x_1,\ ...,\ x_{\frac{k}{2}+1} = -x_{\frac{k}{2}-1}.$$

So,

$$\begin{aligned}\pi(x) &= \prod_{i=1}^{k}(x - x_i)\\ &= (x - x_0)(x - x_1)...(x - x_{\frac{k}{2}-1})(x - x_{\frac{k}{2}})(x - x_{\frac{k}{2}+1})...(x - x_{k-1})(x - x_k)\\ &= (x - x_0)(x - x_1)...(x - x_{\frac{k}{2}-1})(x - 0)(x + x_{\frac{k}{2}-1})...(x + x_1)(x + x_0)\\ &= (x^2 - x_0^2)(x^2 - x_1^2)...(x^2 - x_{\frac{k}{2}-1}^2)x, \quad |x| \le 1.\end{aligned}$$

Hence,

$$|\pi(x)| \le \left|(1 - x_0^2)(1 - x_1^2)...(1 - x_{\frac{k}{2}-1}^2)x\right|. \tag{4.2}$$

Similar to the case of odd k, we have

$$0 < 1 - x_i^2 < 1 \quad \text{for} \quad i = 0, 1, ..., k/2 - 1.$$

Consequently, from (4.2) we obtain $|\pi(x)| < 1$. □

The following theorem discusses the convergence of the approximate solution obtained from the prescribed method in Section 3.

Theorem 3. *Suppose K and ϕ in (3.1) satisfy $K \in C([-1,1]^2)$ and $\phi \in C^{k+1}[-1,1]$ with $k \geq 0$. If $u(x)$, the exact solution, and $u_k(x)$, the approximate solution defined by (3.10), are both in $C^{k+1}[-1,1]$, then $u_k(x)$ is uniformly convergent to $u(x)$.*

Proof. Since $\phi, u \in C^{k+1}[-1,1]$, there is some $M > 0$ with $|\Phi^{(k+1)}(x)| \leq M$ for all $x \in [-1,1]$, where $\Phi(x) = \phi(u(x))$. Thus using (3.2) and (3.3) we may proceed as follows

$$\begin{aligned}\|u(x) - u_k(x)\|_E^2 &= \int_{-1}^{1} |u(x) - u_k(x)|^2 dx \\ &= \int_{-1}^{1} \left| \int_{-1}^{1} K(x,t)\left(\Phi(t) - \Phi_k(t)\right) dt \right|^2 dx \leq \int_{-1}^{1} \left(\int_{-1}^{1} |K(x,t)|^2 dt \right) \\ &\quad \times \left(\int_{-1}^{1} |\Phi(t) - \Phi_k(t)|^2 dt \right) dx = \left(\int_{-1}^{1} \int_{-1}^{1} |K(x,t)|^2 dt dx \right) \\ &\quad \times \left(\int_{-1}^{1} |\Phi(t) - \Phi_k(t)|^2 dt \right) = \|K\|^2 \int_{-1}^{1} |\Phi(t) - \Phi_k(t)|^2 dt. \end{aligned} \tag{4.3}$$

Since $\Phi_k(t)$ is the Lagrange interpolation polynomial for $\Phi(t)$, according to Theorem 1 we have

$$\Phi(t) - \Phi_k(t) = \frac{\pi(t)}{(k+1)!}\Phi^{(k+1)}(\zeta), \ \zeta \in [-1,1]. \tag{4.4}$$

Taking into account the bounds for the functions $|\pi(x)|$ and $|\Phi^{(k+1)}(x)|$, (4.4) yields

$$|\Phi(t) - \Phi_k(t)| \leq \frac{M}{(k+1)!}. \tag{4.5}$$

Consequently by substituting (4.5) into (4.3) we observe that

$$\|u(x) - u_k(x)\|_E^2 \leq \|K\|^2 \int_{-1}^{1} \left(\frac{M}{(k+1)!} \right)^2 dt = 2 \left(\frac{\|K\| M}{(k+1)!} \right)^2.$$

Hence,

$$\|u(x) - u_k(x)\|_E \leq \frac{C}{(k+1)!}, \quad C = \sqrt{2}\|K\|M. \tag{4.6}$$

Now we show that for every $\varepsilon > 0$ there exists a positive integer N such that

$$k \geq N \Rightarrow \|u(x) - u_k(x)\|_E < \varepsilon, \qquad \text{for all } \ x \in [-1,1].$$

For every $\varepsilon > 0$, set $N = \frac{C}{\varepsilon} > 0$. Now if $k \geq N$, then

$$\frac{C}{(k+1)!} < \frac{C}{k} \leq \frac{C}{N} = \varepsilon.$$

Therefore (4.6) gives

$$\|u(x) - u_k(x)\|_E < \varepsilon, \qquad \text{for all } \ x \in [-1,1];$$

that is, $u_k(x)$ is uniformly convergent to $u(x)$. □

Theorem 4. *Consider Equation* (3.1) *under the hypotheses of Theorem* 3. *Then the obtained solution from the prescribed method in Section 3 is stable.*

Proof. Suppose $f(x)$ has $\varepsilon(x) > 0$ perturbation where we assume $\varepsilon(x)$ is bounded on $[-1, 1]$. Let $\tilde{\varepsilon} = \sup_{|x|\leq 1} \varepsilon(x)$. We show that the variation of the obtained solution is bounded by a constant multiple of ε.

Take $u_k^1(x)$ as the approximated solution of Equation (3.1) which is of the form (3.10), that is

$$u_k^1(x) = f(x) + \int_{-1}^{1} K(x,t)\Phi_k(t)dt = f(x) + \sum_{i=0}^{k} \alpha_i \omega_i K(x,t_i).$$

Also, let $u_k^2(x)$ be the approximated solution of

$$u(x) = f(x) + \varepsilon(x) + \int_{-1}^{1} K(x,t)\phi(u(t))dt, \quad |x| < 1,$$
$$u_k^2(x) = f(x) + \varepsilon(x) + \int_{-1}^{1} K(x,t)\Phi_k(t)dt.$$

Analogous to the Equation (3.4), we have

$$\begin{aligned}\Phi_k(t) &= \phi\left(f(t) + \varepsilon(t) + \int_{-1}^{1} K(t,x)\Phi_k(x)dx\right)\\ &= \phi(\tau(t)) + \phi'(\tau(t))\varepsilon(t) + \phi''(\tau(t))\frac{(\varepsilon(t))^2}{2!} + \phi'''(\tau(t))\frac{(\varepsilon(t))^3}{3!} + \cdots, \end{aligned} \tag{4.7}$$

where $\tau(t) = f(t) + \int_{-1}^{1} K(t,x)\Phi_k(x)dx$. We observe that

$$\phi(\tau(t)) = \phi\left(f(t) + \int_{-1}^{1} K(t,x)\Phi_k(x)dx\right) = \phi(u_k(t)) = \Phi_k(t),$$

which implies

$$f(x) + \int_{-1}^{1} K(x,t)\phi(\tau(t))dt = f(x) + \int_{-1}^{1} K(x,t)\Phi_k(t)dt = u_k^1(x). \tag{4.8}$$

Now by substituting (4.7) into $u_k^2(x)$ and using (4.8) we can write

$$\begin{aligned} u_k^2(x) &= f(x) + \varepsilon(x) + \int_{-1}^{1} K(x,t)\Phi_k(t)dt = f(x) + \varepsilon(x)\\ &+ \int_{-1}^{1} K(x,t)\left(\phi(\tau(t)) + \phi'(\tau(t))\varepsilon(t) + \phi''(\tau(t))\frac{(\varepsilon(t))^2}{2!}\right.\\ &\left.+\phi'''(\tau(t))\frac{(\varepsilon(t))^3}{3!} + \ldots\right)dt = \left(f(x) + \int_{-1}^{1} K(x,t)\phi(\tau(t))dt\right)\end{aligned}$$

$$+\varepsilon(x)+\int_{-1}^{1}K(x,t)\phi'(\tau(t))\varepsilon(t)dt+\frac{1}{2!}\int_{-1}^{1}K(x,t)\phi''(\tau(t))(\varepsilon(t))^2dt$$
$$+\frac{1}{3!}\int_{-1}^{1}K(x,t)\phi'''(\tau(t))(\varepsilon(t))^3dt+\ldots=u_k^1(x)+\varepsilon(x)$$
$$+\int_{-1}^{1}K(x,t)\phi'(\tau(t))\varepsilon(t)dt+\frac{1}{2!}\int_{-1}^{1}K(x,t)\phi''(\tau(t))(\varepsilon(t))^2dt$$
$$+\frac{1}{3!}\int_{-1}^{1}K(x,t)\phi'''(\tau(t))(\varepsilon(t))^3dt+\ldots=u_k^1(x)+r(x), \tag{4.9}$$

where

$$r(x)=\varepsilon(x)+\int_{-1}^{1}K(x,t)\phi'(\tau(t))\varepsilon(t)dt+\frac{1}{2!}\int_{-1}^{1}K(x,t)\phi''(\tau(t))(\varepsilon(t))^2dt$$
$$+\frac{1}{3!}\int_{-1}^{1}K(x,t)\phi'''(\tau(t))(\varepsilon(t))^3dt+\ldots. \tag{4.10}$$

Let $|K(x,t)|\le M_1$ for all $(x,t)\in[-1,1]^2$ and $|\phi^{(i)}(\tau(t))|\le M_2$ for all $t\in[-1,1]$ and $i>0$. Thus (4.10) gives

$$\begin{aligned}|r(x)|\le&\varepsilon(x)+\int_{-1}^{1}|K(x,t)||\phi'(\tau(t))|\varepsilon(t)dt+\frac{1}{2!}\int_{-1}^{1}|K(x,t)||\phi''(\tau(t))|(\varepsilon(t))^2dt\\&+\frac{1}{3!}\int_{-1}^{1}|K(x,t)||\phi'''(\tau(t))|(\varepsilon(t))^3dt+\ldots\\&\le\tilde{\varepsilon}+2M_1M_2\tilde{\varepsilon}+2M_1M_2\frac{\tilde{\varepsilon}^2}{2!}+2M_1M_2\frac{\tilde{\varepsilon}^3}{3!}+\ldots\\&=\tilde{\varepsilon}+2M_1M_2\left(\tilde{\varepsilon}+\frac{\tilde{\varepsilon}^2}{2!}+\frac{\tilde{\varepsilon}^3}{3!}+\ldots\right)=\tilde{\varepsilon}+2M_1M_2\left(e^{\tilde{\varepsilon}}-1\right).\end{aligned} \tag{4.11}$$

Now for every $\bar{\varepsilon}>0$ there exists $\delta>0$ such that, $0<\tilde{\varepsilon}<\delta\ \Rightarrow\ |e^{\tilde{\varepsilon}}-1|<\bar{\varepsilon}$. Thus (4.11) gives

$$|r(x)|<\tilde{\varepsilon}+2M_1M_2\bar{\varepsilon}.$$

Setting $\varepsilon=\max\{\tilde{\varepsilon},\bar{\varepsilon}\}$ necessitates that

$$|r(x)|<(1+2M_1M_2)\varepsilon.$$

Consequently, from (4.9) we obtain

$$|u_k^2(x)-u_k^1(x)|^2=|r(x)|^2<(1+2M_1M_2)^2\varepsilon^2. \tag{4.12}$$

Integrating both sides of (4.12) over $[-1,1]$ gives

$$\|u_k^2-u_k^1\|_E^2=\int_{-1}^{1}|u_k^2(x)-u_k^1(x)|^2dx<2(1+2M_1M_2)^2\varepsilon^2.$$

So,

$$\|u_k^2-u_k^1\|_E<C\varepsilon,\quad C=\sqrt{2}(1+2M_1M_2),$$

which means stability of the method. □

5 Numerical examples

In this section, we apply the method proposed in Section 3, to some test examples. All numerical calculations are performed by Maple 13.

Example 1. [12, 33]

$$u(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)\,(u(t))^3\,dt, \quad 0 \le x \le 1,$$

with exact solution $u(x) = \sin(\pi x) + \frac{1}{3}(20 - \sqrt{391})\cos(\pi x)$. The absolute error $|u_k(x) - u(x)|$ of Example 1 for $k = 2, 3, 4, 7$ is shown in Table 1.

Table 1. Absolute error of Example 1.

x	$k=2$	$k=3$	$k=4$	$k=7$	[33] (N=16)	[12] (n=25 and m=9)
0.0	1.5e−2	3.2e−3	4.1e−4	9.5e−8	1.1e−3	−−
0.1	1.4e−2	3.0e−3	3.9e−4	9.2e−8	1.9e−4	4.4e−10
0.2	1.2e−2	2.6e−3	3.3e−4	7.8e−8	21.5e−3	3.7e−10
0.3	9.0e−3	1.9e−3	2.4e−4	5.4e−8	5.6e−4	2.7e−10
0.4	4.7e−3	1.0e−3	1.2e−4	2.9e−8	1.2e−3	1.4e−10
0.5	0.0	0.0	0.0	0.0	1.3e−3	0.0
0.6	4.7e−3	1.0e−3	1.2e−4	3.0e−8	1.2e−3	1.4e−10
0.7	9.0e−3	1.9e−3	2.4e−4	5.5e−8	3.0e−4	2.7e−10
0.8	1.2e−2	2.6e−3	3.3e−4	7.6e−8	1.3e−3	3.7e−10
0.9	1.4e−2	3.0e−3	3.9e−4	9.3e−8	4.9e−5	3.3e−10
1.0	1.5e−3	3.2e−3	4.1e−4	9.5e−8	6.2e−2	−−

Example 2. [12]

$$u(x) = \int_0^1 xt\sqrt{u(t)}dt + 2 - \frac{1}{3}(2\sqrt{2} - 1)x - x^2, \quad 0 \le x \le 1,$$

with exact solution $u(x) = 2 - x^2$. The absolute errors $|u_k(x) - u(x)|$ of Example 2 for $k = 2, 3, 4, 7$ are shown in Table 2.

Table 2. Absolute error of Example 2.

x	$k=2$	$k=3$	$k=4$	$k=7$	[12] (n=20 and m=9)
0.1	5.9e−6	3.1e−7	1.8e−8	1.4e−10	1.4e−12
0.3	1.5e−5	9.5e−7	5.8e−8	4.0e−10	4.4e−12
0.5	2.9e−5	1.5e−6	9.7e−8	4.0e−10	7.4e−12
0.7	4.1e−5	2.2e−6	1.7e−7	0.0	1.0e−11
0.9	5.9e−5	2.8e−6	1.7e−7	8.0e−10	1.3e−11

Example 3. [22]

$$u(x) = \left(\frac{1}{2} - \ln(2)\right)x^2 + \sqrt{x} + \int_0^1 \frac{x^2t^2}{1 + u^2(t)}dt, \quad 0 \le x \le 1,$$

Table 3. Absolute error of Example 3.

x	$k=2$	$k=3$	$k=4$	$k=7$	[22] ($N=5$)	[22] ($N=25$)
0.05	$5.7e{-8}$	$1.7e{-9}$	$5.1e{-11}$	$4.2e{-14}$	$8.3e{-12}$	$5.6e{-16}$
0.1	$2.2e{-7}$	$6.8e{-9}$	$2.0e{-10}$	$1.7e{-13}$	$3.3e{-11}$	$2.2e{-15}$
0.25	$1.4e{-6}$	$4.3e{-8}$	$1.2e{-9}$	$1.0e{-12}$	$2.8e{-10}$	$1.4e{-14}$
0.5	$5.7e{-6}$	$1.7e{-7}$	$5.1e{-9}$	$4.2e{-12}$	$8.3e{-10}$	$5.6e{-14}$
0.75	$1.2e{-5}$	$3.8e{-7}$	$1.1e{-8}$	$9.6e{-12}$	$1.8e{-9}$	$1.2e{-13}$
1.0	$2.2e{-5}$	$6.8e{-7}$	$2.0e{-8}$	$1.7e{-11}$	$3.3e{-9}$	$2.2e{-13}$

with exact solution $u(x)=\sqrt{x}$. The absolute errors $|u_k(x)-u(x)|$ of Example 3 for $k=2,3,4,7$ are shown in Table 3.

Example 4. [28]

$$u(x)=e^{x-1}(e-e^2+1)+\int_{-1}^{1}e^{x-2t}\left(u(t)\right)^3dt,\quad |x|\leq 1,$$

with exact solution $u(x)=e^x$. The absolute errors $|u_k(x)-u(x)|$ of Example 4 for $k=2,3,4$ are shown in Table 4.

Table 4. Absolute error of Example 4.

x	$k=2$	$k=3$	$k=4$	method of [28] with ($N=6$)
-1.0	$3.9e{-6}$	$1.8e{-8}$	$2.3e{-10}$	$7.1e{-6}$
-0.8	$4.8e{-6}$	$2.2e{-8}$	$7.4e{-10}$	$1.8e{-6}$
-0.6	$5.9e{-6}$	$2.5e{-8}$	$2.2e{-10}$	$2.3e{-6}$
-0.4	$7.2e{-6}$	$3.2e{-8}$	0.0	$1.8e{-8}$
-0.2	$8.8e{-6}$	$3.9e{-8}$	$6.0e{-10}$	$2.2e{-6}$
0.0	$1.0e{-5}$	$4.8e{-8}$	0.0	$1.3e{-7}$
0.2	$1.3e{-5}$	$5.8e{-8}$	$4.4e{-10}$	$2.2e{-6}$
0.4	$1.6e{-5}$	$7.1e{-8}$	$5.4e{-10}$	$2.5e{-7}$
0.6	$1.9e{-5}$	$8.9e{-8}$	$1.3e{-9}$	$2.5e{-6}$
0.8	$2.4e{-5}$	$1.0e{-8}$	0.0	$1.8e{-6}$
1.0	$2.9e{-5}$	$1.3e{-7}$	$1.0e{-9}$	$8.1e{-6}$

For this example, the exact and approximated solutions are compared in Figure 1 with $k=7$. In addition, the absolute error function $|u(x)-u_k(x)|$ is plotted in Figure 2 with the same value for k.

Example 5. (Constructed by authors)

$$u(x)=\cos(x)-x\left(-\frac{1}{2}\sin(1)\cos^3(1)+\frac{1}{4}\cos(1)\sin(1)+\frac{1}{4}\right)+\int_{-1}^{1}x\sin^2(t)\left(u(t)\right)^2dt,\quad |x|\leq 1.$$

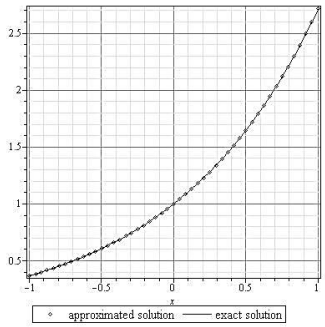

Figure 1. Exact and approximated solutions of Example 4 ($k = 7$).

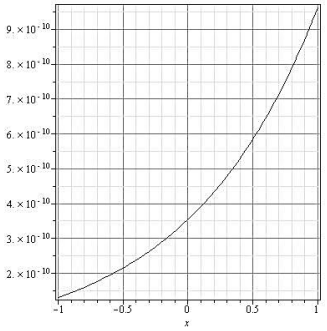

Figure 2. Absolute error of Example 4 ($k = 7$).

To calculate the error in the interval $[-1, 1]$, we define the error function $e_k(x)$ as

$$e_k(x) = u_k(x) - f(x) - \int_{-1}^{1} K(x,t)(u_k(t))^2 dt.$$

In fact, in the right hand side of above equation, we put the approximate solution $u_k(x)$ instead of the exact solution $u(x)$ of Equation (3.1). Now the absolute errors $|e_k(x)|$ of Example 5 for $k = 2, 3, 4, 7$ and some $x \in [-1, 1]$ are shown in Table 5. The exact solution of this example is $u(x) = \cos(x)$.

The exact and approximated solutions are compared in Figure 3 with $k = 7$. In addition, the absolute error function $|e_k(x)|$ is plotted in Figure 4 with the same value for k.

Table 5. Absolute error of Example 5.

x	$e_2(x)$	$e_3(x)$	$e_4(x)$	$e_7(x)$
-1.0	$1.1e{-2}$	$1.5e{-3}$	$7.5e{-5}$	$9.0e{-10}$
-0.8	$1.1e{-2}$	$1.2e{-3}$	$6.0e{-5}$	$8.0e{-10}$
-0.6	$1.1e{-2}$	$9.4e{-4}$	$4.5e{-5}$	$6.0e{-10}$
-0.4	$1.1e{-2}$	$6.3e{-4}$	$3.0e{-5}$	$5.0e{-10}$
-0.2	$1.1e{-2}$	$3.1e{-4}$	$1.5e{-5}$	$2.0e{-10}$
0.0	$1.1e{-2}$	0.0	0.0	0.0
0.2	$1.1e{-2}$	$3.1e{-4}$	$1.5e{-5}$	$2.0e{-10}$
0.4	$1.1e{-2}$	$6.3e{-4}$	$3.0e{-5}$	$5.0e{-10}$
0.6	$1.1e{-2}$	$9.4e{-4}$	$4.5e{-5}$	$6.0e{-10}$
0.8	$1.1e{-2}$	$1.2e{-3}$	$6.0e{-5}$	$8.0e{-10}$
1.0	$1.1e{-2}$	$1.5e{-3}$	$7.5e{-5}$	$9.0e{-10}$

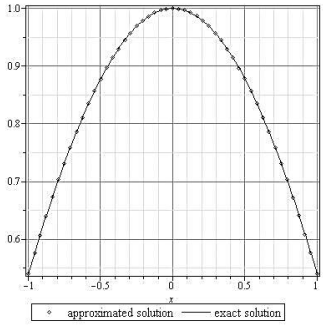

Figure 3. Exact and approximated solutions of Example 5 ($k = 7$).

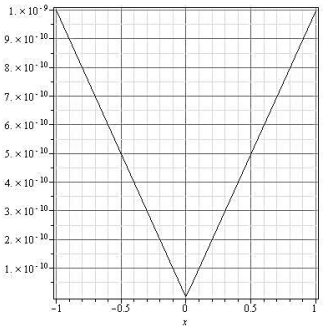

Figure 4. Absolute error of Example 5 ($k = 7$).

Example 6. [14,26,29] Consider the following Lichtenstein-Gershgorin integral equation for $\nu = 1.2$, and $-1 \le x \le 1$,

$$u(x) = 2\tan^{-1}\left(\frac{\nu\sin(\pi x)}{\nu^2(\cos(\pi x)+\cos^2(\pi x))+\sin^2(\pi x)}\right) + \int_{-1}^{1}\frac{\nu u(t)}{(\nu^2+1)-(\nu^2-1)\cos(\pi(x+t))}dt.$$

This equation has been used for the determination of conformal mapping of an ellipse onto a circle as mentioned in [14]. Table 6 gives a comparison between the approximated solutions obtained by the presented method for $k = 3, 4, 7$ and the methods proposed in [14, 26, 29].

Table 6. Numerical results of Example 6.

x	$k=3$	$k=4$	$k=7$	[14] (n=10)	[29] (n=10)	[26] (n=10)
0.1	0.2151	0.2147	0.2155	0.215	0.2158	0.2158
0.2	0.4623	0.4599	0.4618	0.461	0.4625	0.4625
0.3	0.7696	0.7628	0.7663	0.766	0.7673	0.7679
0.4	1.1497	1.1372	1.1421	1.142	1.1434	1.1434
0.5	1.5826	1.5652	1.5705	1.570	1.5721	1.5721
0.6	2.0136	1.9943	1.9991	1.999	2.0007	2.0007
0.7	2.3891	2.3716	2.3750	2.375	2.3765	2.3767
0.8	2.6904	2.6777	2.6796	2.679	2.6808	2.6808
0.9	2.9317	2.9252	2.9260	2.926	2.9266	2.9265

Example 7. [6,32] Consider the mathematical model for an adiabatic tubular chemical reactor which in the case of steady state solutions, can be stated as the ordinary differential equation

$$u''(x) - \lambda u'(x) + F(\lambda,\mu,\beta,u(x)) = 0, \quad x \in [0,1],$$

with boundary conditions

$$u^{'}(0) = \lambda u(0), \quad u^{'}(1) = 0,$$

where $F(\lambda, \mu, \beta, u(x)) = \lambda\mu(\beta - u(x))e^{u(x)}$. The problem can be converted into a Hammerstein integral equation of the form [25]

$$u(x) = \int_0^1 \mu\kappa(x,t)(\beta - u(t))e^{u(t)}dt, \quad x \in [0,1], \tag{5.1}$$

where

$$\kappa(x,t) = \begin{cases} 1, & t \le x, \\ e^{\lambda(x-t)}, & x \le t. \end{cases}$$

The existence and uniqueness of the solution for the above Hammerstein integral equation with respect to the values of parameters λ, μ and β are discussed in [25]. Integral equation (5.1) is solved in [6, 32] for the particular values of the parameters $\lambda = 10$, $\mu = 0.02$ and $\beta = 3$. Table 7 gives a comparison between the approximated solutions obtained by the presented method for $k = 4, 7$ and the methods proposed in [6, 32].

Table 7. Numerical results of Example 7.

x	$k = 4$	$k = 7$	[6] (N=10)	[32] (M=4)
0.0	0.006051	0.006050	0.006048	0.006048
0.2	0.018799	0.018011	0.018192	0.018193
0.4	0.028583	0.031527	0.030424	0.030424
0.6	0.042223	0.043986	0.042669	0.042669
0.8	0.055762	0.054519	0.054371	0.054368
1.0	0.061564	0.061505	0.061459	0.061505

6 Conclusions

In this work, we have proposed a simple and accurate method for evaluating the solution of the nonlinear Fredholm integral equation which is based on the use of the Lagrange interpolation simultaneously with the Gauss-Legendre integration. Implementation of the method demonstrates this method is effective and computationally attractive. The key point here, is to choose the Lagrange interpolation points to be the same as the points for the Gauss quadrature formula and transforming the main problem to a system of nonlinear algebraic equations that can be solved by Newton's method. The approximate solution is uniformly convergent to the exact solution. Furthermore, the solution of the proposed method is stable. Some test examples are carried out which confirm effectiveness of the proposed method. It is stated that high accuracy is achieved even by using a few number of interpolation polynomials.

We suggest this method to the Volterra-Hammerstein integral equations. The method can be extended to the system of Fredholm and Volterra Hammerstein integral equations through some modifications. Also, other types of

numerical integration methods and interpolation functions can be examined for the same type of integral equations.

References

[1] C. Allouch, D. Sbibih and M. Tahrichi. Legendre superconvergent Galerkin-collocation type methods for Hammerstein equations. *Journal of Computational and Applied Mathematics*, **353**:253–264, 2019. https://doi.org/10.1016/j.cam.2018.12.040.

[2] H. Almasieh and M. Roodaki. Triangular functions method for the solution of Fredholm integral equations system. *Ain Shams Engineering Journal*, **3**(4):411–416, 2012. https://doi.org/10.1016/j.asej.2012.04.006.

[3] P. Assari. A meshless local Galerkin method for the numerical solution of Hammerstein integral equations based on the moving least squares technique. *Journal of Applied Analysis & Computation*, **9**(1):75–104, 2019. https://doi.org/10.11948/2019.75.

[4] P. Assari. The numerical solution of Fredholm-Hammerstein integral equations by combining the collocation method and radial basis functions. *Filomat*, **33**(3):667–682, 2019. https://doi.org/10.2298/FIL1903667A.

[5] E. Babolian and A. Shahsavaran. Numerical solution of nonlinear Fredholm integral equations of the second kind using haar wavelets. *Journal of Computational and Applied Mathematics*, **1**(225):87–95, 2009. https://doi.org/10.1016/j.cam.2008.07.003.

[6] S. Bazm. Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations. *Journal of Computational and Applied Mathematics*, **275**:44–60, 2015. https://doi.org/10.1016/j.cam.2014.07.018.

[7] H. Beyrami, T. Lotfi and K. Mahdiani. A new efficient method with error analysis for solving the second kind Fredholm integral equations with Cauchy kernel. *Journal of Computational and Applied Mathematics*, **300**(C):385–399, 2016. https://doi.org/10.1016/j.cam.2016.01.011.

[8] M.C. Bounaya, S. Lemita, M. Ghiat and M.Z. Aissaoui. On a nonlinear integro-differential equation of Fredholm type. *International Journal of Computing Science and Mathematics*, **13**(2), 2021. https://doi.org/10.1504/IJCSM.2021.114188.

[9] V. Cristini, J. Lowengrub and Q. Nie. Nonlinear simulation of tumor growth. *Journal of Mathematical Biology*, **46**:191–224, 2003. https://doi.org/10.1007/s00285-002-0174-6.

[10] A. Daddi-Moussa-Ider, B. Kaoui and H. Löwen. Axisymmetric flow due to a Stokeslet near a finite-sized elastic membrane. *Journal of the Physical Society of Japan*, **88**(5), 2019. https://doi.org/10.7566/JPSJ.88.054401.

[11] L.M. Delves and J.L. Mohamed. *Computational Methods for Integral Equations*. Cambridge University Press, 1985. https://doi.org/10.1017/CBO9780511569609.

[12] M. Derakhshan and M. Zarebnia. On the numerical treatment and analysis of Hammerstein integral equation. *Computational Methods for Differential Equations*, **9**(2):493–510, 2021. https://doi.org/10.22034/cmde.2019.29825.1435.

[13] S.A. Edalatpanah. Legendre wavelets collocation method for systems of linear and nonlinear Fredholm integral equations. *Aloy Journal of Soft Computing and Applications*, **2**(1):33–44, 2014.

[14] D. Elliott. A Chebyshev series method for the numerical solution of Fredholm integral equations. *The Computer Journal*, **6**(1):102–112, 1963. https://doi.org/10.1093/comjnl/6.1.102.

[15] R. Ezzati and S. Ziari. Numerical solution of nonlinear fuzzy Fredholm integral equations using iterative method. *Applied Mathematics and Computation*, **225**:33–42, 2013. https://doi.org/10.1016/j.amc.2013.09.020.

[16] F. Fattahzadeh. Numerical solution of general nonlinear Fredholm-Volterra integral equations using Chebyshev approximation. *International Journal Industrial Mathematics*, **8**(1):81–86, 2016.

[17] L. Fermo. A Nyström method for a class of Fredholm integral equations of the third kind on unbounded domains. *Applied Numerical Mathematics*, **59**(12):2970–2989, 2009. https://doi.org/10.1016/j.apnum.2009.07.002.

[18] G. He, S. Xiang and Z. Xu. A Chebyshev collocation method for a class of Fredholm integral equations with highly oscillatory kernels. *Journal of Computational and Applied Mathematics*, **300**:354–368, 2016. https://doi.org/10.1016/j.cam.2015.12.027.

[19] M.A. Hernández-Verón and E. Martínez. On nonlinear Fredholm integral equations with non-differentiable Nemystkii operator. *Mathematical Methods in the Applied Sciences*, **43**(14):7961–7976, 2020. https://doi.org/10.1002/mma.5801.

[20] K.G.T. Hollands. The simplified-Fredholm integral equation solver and its use in thermal radiation. *ASME Journal of Heat Transfer*, **132**(2):1–6, 2010. https://doi.org/10.1115/1.4000183.

[21] N. Karamollahi, M. Heydari and G.B. Loghmani. Approximate solution of nonlinear Fredholm integral equations of the second kind using a class of Hermite interpolation polynomials. *Mathematics and Computers in Simulation*, **187**:414–432, 2021. https://doi.org/10.1016/j.matcom.2021.03.015.

[22] R. Katani. Numerical solution of the Fredholm integral equations with a quadrature method. *SeMA Journal*, **76**(2):271–276, 2019. https://doi.org/10.1007/s40324-018-0175-z.

[23] W.V. Lovitt. *Linear Integral Equations*. Dover, New York, 1950.

[24] Y. Lu, L. Shen and Y. Xu. Integral equation models for image restoration: high accuracy methods and fast algorithms. *Inverse Problems*, **26**(4):1–32, 2010. https://doi.org/10.1088/0266-5611/26/4/045006.

[25] N. Madbouly. *Solutions of Hammerstein integral equations arising from chemical reactor theory (Ph.D. thesis)*. University of Strathclyde, 1996.

[26] K. Maleknejad, J. Rashidinia and H. Jalilian. Quintic spline functions and Fredholm integral equation. *Computational Methods for Differential Equations*, **9**(1):211–224, 2021. https://doi.org/10.22034/cmde.2019.31983.1492.

[27] A. Molabahrami. A modified degenerate kernel method for the system of Fredholm integral equations of the second kind. *Iranian Journal of Mathematical Sciences and Informatics*, **14**(1):43–53, 2019.

[28] H. Saberi Najafi, S.A. Edalatpanah and H. Aminikhah. An algorithmic approach for solution of nonlinear Fredholm-Hammerstein integral equations. *Iranian Journal Science and Technology*, **39**(3):399–406, 2015. https://doi.org/10.22099/IJSTS.2015.3263.

[29] S. Panda, S.C. Martha and A. Chakrabarti. A modified approach to numerical solution of Fredholm integral equations of the second kind. *Applied Mathematics and Computation*, **271**:102–112, 2015. https://doi.org/10.1016/j.amc.2015.08.111.

[30] B.L. Panigrahi, M. Mandal and G. Nelakanti. Legendre multi-Galerkin methods for Fredholm integral equations with weakly singular kernel and the corresponding eigenvalue problem. *Journal of Computational and Applied Mathematics*, **346**:224–236, 2019. https://doi.org/10.1016/j.cam.2018.07.010.

[31] A. Rahmoune. Spectral collocation method for solving Fredholm integral equations on the half-line. *Applied Mathematics and Computation*, **219**(17):9254–9260, 2013. https://doi.org/10.1016/j.amc.2013.03.043.

[32] P.K. Sahu, A.K. Ranjan and S.S. Ray. B-spline wavelet method for solving Fredholm Hammerstein integral equation arising from chemical reactor theory. *Nonlinear Engineering*, **7**(3):163–169, 2018. https://doi.org/10.1515/nleng-2017-0116.

[33] M.H.A. Sathar, A.F.N. Rasedee, A.A. Ahmedov and N. Bachok. Numerical solution of nonlinear Fredholm and Volterra integrals by Newton-Kantorovich and haar wavelets methods. *Symmetry*, **12**(12), 2020. https://doi.org/10.3390/sym12122034.

[34] G. Spiga, R.L. Bowden and V.C. Boffi. On the solution to a class of nonlinear integral equations arising in transport theory. *Journal of Mathematical Physics*, **25**(12):3444–3450, 1984. https://doi.org/10.1063/1.526099.

[35] J. Stoer and R. Bulirsch. *Introduction to numerical analysis*. Springer-Verlag, 1991.

[36] P.N. Swarztrauber. On computing the points and weights for Gauss-Legendre quadrature. *SIAM Journal on Scientific Computing*, **24**(8):945–954, 2003. https://doi.org/10.1137/S1064827500379690.

[37] A.M. Wazwaz. *Linear and nonlinear integral equations: methods and applications*. Higher education, Springer, 2011.

[38] M.A. Zaky and A.S. Hendy. Convergence analysis of a Legendre spectral collocation method for nonlinear Fredholm integral equations in multidimensions. *Mathematical Methods in the Applied Sciences*, 2020. https://doi.org/10.1002/mma.6443.