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Universality of Zeta-Functions of Cusp Forms and Non-Trivial Zeros of the Riemann Zeta-Function

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Abstract. It is known that zeta-functions $\zeta(s,F)$ of normalized Hecke-eigen cusp forms F are universal in the Voronin sense, i.e., their shifts $\zeta(s+i\tau,F)$, $\tau\in\mathbb{R}$, approximate a wide class of analytic functions. In the paper, under a weak form of the Montgomery pair correlation conjecture, it is proved that the shifts $\zeta(s+i\gamma_k h,F)$, where $\gamma_1<\gamma_2<\dots$ is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and $h>0$, also approximate a wide class of analytic functions.

Keywords: Montgomery pair correlation conjecture, Riemann zeta-function, zeta-function of cusp form, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

We start with some definitions. Let

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}$$

be the full modular group. Suppose that $F(z)$ is a holomorphic function in the upper half-plane, and, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z)$$

for some $\kappa \in 2\mathbb{N}$. Then $F(z)$ has the Fourier series expansion at infinity

$$F(z) = \sum_{m=-\infty}^{\infty} c(m) \mathrm{e}^{2\pi i m z}.$$

If $c(m)=0$ for $m<0$, then $F(z)$ is called a modular form of weight κ. If the modular form $F(s)$ has the Fourier series expansion at infinity

$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi i m z},$$

then it is called a cusp form of weight κ for the full modular group.

Suppose that $F(z)$ is a cusp form of weight κ for the full modular group. Then the zeta-function

$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad s=\sigma+it$$

can be attached to $F(z)$. The latter series, in view of the estimate

$$c(m) \ll m^{\frac{\kappa-1}{2}},$$

is absolutely convergent for $\sigma > \frac{\kappa+1}{2}$. Moreover, it has analytic continuation to an entire function.

We additionally require that the function $F(z)$ would be the Hecke-eigen cusp form, i.e., that $F(z)$ would be the eigenfunction of all Hecke operators T_m,

$$T_m f(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^{\kappa}} \sum_{b(\mathrm{mod} d)} F\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N}.$$

Then the form $F(z)$ can be normalized, thus, we may suppose that $c(1)=1$.

In the sequel, we suppose that $F(z)$ is a normalized Hecke-eigen cusp form of weight κ. In this case, the zeta-function $\zeta(s,F)$ has, for $\sigma > \frac{\kappa+1}{2}$, the Euler product representation over primes

$$\zeta(s,F) = \prod_p \left(1-\alpha(p)/p^s\right)^{-1} \left(1-\beta(p)/p^s\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $\alpha(p) + \beta(p) = c(p)$.

In [8], it was proved that the function $\zeta(s, F)$ is universal in the Voronin sense, i.e., a wide class of analytic functions is approximated by shifts $\zeta(s + i\tau, F)$, $\tau \in \mathbb{R}$. More precisely, let $D_\kappa = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$. Denote by $\mathcal{K}_F$ the class of compact subsets of the strip D_κ with connected complements, and by $H_{0F}(K)$ with $K \in \mathcal{K}_F$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the main result of [8] is the following statement.

Theorem 1. *Suppose that $K \in \mathcal{K}_F$ and $f(s) \in H_{0F}(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon\right\} > 0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

In the latter theorem, τ in shifts $\zeta(s + i\tau, F)$ takes arbitrary real values, therefore, the theorem is of continuous type. Also, Theorem 1 has a discrete version when τ in $\zeta(s + i\tau, F)$ takes values from certain discrete sets. The classical discrete set is an arithmetical progression $\{kh : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, where $h > 0$ is a fixed number. Discrete universality theorems for the function $\zeta(s, F)$ were considered in [9] and [10], and the following statement has been obtained.

Theorem 2. *Suppose that $K \in \mathcal{K}_F$, $f(s) \in H_{0F}(K)$ and $h > 0$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \le k \le N : \sup_{s\in K} |\zeta(s + ikh, F) - f(s)| < \varepsilon\right\} > 0.$$

Here $\#A$ denotes the cardinality of a set A, and N runs over non-negative integers.

In [12], more general shifts $\zeta(s + i\varphi(k), F)$ were used. Here $\varphi(t)$ is a real-valued positive increasing function on $[k_0 - \frac{1}{2}, \infty)$, $k_0 \in \mathbb{N}$, having a continuous derivative $\varphi'(t)$ satisfying the estimate

$$\varphi(2t) \max_{t\le u\le 2t} \left(\frac{1}{\varphi'(u)} + \varphi'(u)\right) \ll t,$$

and such that the sequence $\{a\varphi(k) : k \geqslant k_0\}$ with every $a \in \mathbb{R}\backslash\{0\}$ is uniformly distributed modulo 1.

In [13], a joint version of a theorem from [12] has been proved.

The aim of this paper is an extension of Theorem 2 for the discrete set related to non-trivial zeros of the Riemann zeta-function $\zeta(s)$ which is defined, for $\sigma > 1$, by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and has a meromorphic continuation to the whole complex plane. The function $\zeta(s)$ has infinitely many so-called non-trivial zeros $\varrho=\beta+i\gamma$ lying in the strip $\{s\in\mathbb{C}:0<\sigma<1\}$. By the Riemann hypothesis, all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\frac{1}{2}$.

Thus, let $0<\gamma_1<\gamma_2<\ldots\leqslant\gamma_k\leqslant\ldots$ be the sequence of imaginary parts of non-trivial zeros of the function $\zeta(s)$. We will use a hypothesis on the distribution of the sequence $\{\gamma_k:k\in\mathbb{N}\}$, namely, we suppose that, for $c>0$,

$$\sum_{\substack{\gamma_k\leqslant T\\ |\gamma_k-\gamma_l|<\frac{c}{\log T}}}\sum_{\gamma_l\leqslant T}\ll T\log T,\quad T\to\infty. \tag{1.1}$$

The latter estimate is implied by the famous Montgomery pair correlation conjecture [16]. The main result of the paper is the following theorem.

Theorem 3. *Suppose that the estimate (1.1) is true. Let $K\in\mathcal{K}_F$, $f(s)\in H_{0F}(K)$ and $h>0$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N}\#\left\{1\le k\le N:\sup_{s\in K}|\zeta(s+i\gamma_kh,F)-f(s)|<\varepsilon\right\}>0.$$

Moreover, the limit

$$\lim_{N\to\infty}\frac{1}{N}\#\left\{1\le k\le N:\sup_{s\in K}|\zeta(s+i\gamma_kh,F)-f(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

Theorem 3 with the Riemann hypothesis in place of (1.1) was proved in [4] by using [3].

We recall that the condition (1.1) for the first-time was applied in [5] for the approximation by shifts $\zeta(s+i\gamma_kh)$, and in [7] for joint approximation by shifts $(\zeta(s+i\gamma_kh),\zeta(s+i\gamma_kh,\alpha))$, where $\zeta(s,\alpha)$ is the Hurwitz zeta-function

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},\quad\sigma>1$$

with transcendental parameter α. In [11], the joint approximation by shifts of Dirichlet L-functions involving the sequence $\{\gamma_k\}$ was discussed. Finally, the paper [1] is devoted to a generalization of [7] for shifts of the periodic and periodic Hurwitz zeta-functions.

For the proof of Theorem 3, we will apply some results from [5] and [8]. On the mentioned results, we will construct a probabilistic model.

2 Probabilistic model

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and let $H(D_F)$ be the space of analytic functions on D_F endowed with the topology of uniform convergence on compacta. In this section, we will consider the weak convergence as $N \to \infty$ for

$$P_{N,F}(A) \stackrel{\text{def}}{=} \frac{1}{N}\#\bigl\{1 \le k \le N : \zeta(s+i\gamma_k h, F) \in A\bigr\}, \quad A \in \mathcal{B}(H(D_F)).$$

To state a limit theorem for $P_{N,F}$, we need some notation. Denote by γ the unit circle on the complex plane, by $\mathbb{P}$ the set of all prime numbers, and define the set $\Omega = \prod_{p\in\mathbb{P}} \gamma_p$, where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and operation of pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega = (\omega(p) : p \in \mathbb{P})$ the elements of the torus Ω, and on the above probability space define the $H(D_F)$-valued random element

$$\zeta(s,\omega,F) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

We note that the latter infinite product is uniformly convergent on compact subsets of the strip D_F for almost all $\omega \in \Omega$, thus, it defines an $H(D_F)$-valued random element. Denote by $P_{\zeta,F}$ the distribution of the random element $\zeta(s,\omega,F)$, i.e., for $A \in \mathcal{B}(H(D_F))$,

$$P_{\zeta,F}(A) = m_H\left\{\omega \in \Omega : \zeta(s,\omega,F) \in A\right\}.$$

We will prove the following statement

Theorem 4. *Suppose that the estimate (1.1) is true. Then $P_{N,F}$ converges weakly to the measure $P_{\zeta,F}$ as $N \to \infty$.*

The proof of Theorem 4 consists from three limit theorems that will be stated as separate lemmas.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{N}\#\Bigl\{1 \le k \le \mathrm{N} : \bigl(p^{-i\gamma_k h} : p \in \mathbb{P}\bigr) \in A\Bigr\}.$$

Lemma 1. *Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

The lemma is proved in [5] by using the Fourier transform method. For this, the uniform distribution modulo 1 of the sequence $\{a\gamma_k : k \in \mathbb{N}\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is applied.

The next lemma deals with absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, for $m, n \in \mathbb{N}$, let

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}, \quad \zeta_n(s,F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s}.$$

Then it is known [8] that the latter series is absolutely convergent for $\sigma > \frac{\kappa}{2}$. Consider the mapping $u_{n,F} : \Omega \to H(D_F)$ given by $u_{n,F}(\omega) = \zeta_n(s,\omega,F)$, where

$$\zeta_n(s,\omega,F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)v_n(m)}{m^s}, \quad \omega(m) = \prod_{p^l \mid m,\, p^{l+1} \nmid m} \omega^l(p), \quad m \in \mathbb{N}.$$

Obviously, the series for $\zeta_n(s,\omega,F)$ is also absolutely convergent for $\sigma > \frac{\kappa}{2}$. Therefore, the mapping $u_{n,F}$ is continuous, hence it is $(\mathcal{B}(\Omega),\mathcal{B}(H(D_F)))$-measurable. Therefore, the Haar measure m_H defines the unique probability measure $V_{n,F}=m_H u_{n,F}^{-1}$ on $(H(D_F),\mathcal{B}(H(D_F)))$, where, for $A\in\mathcal{B}(H(D_F))$,

$$V_{n,F}(A)=m_H u_{n,F}^{-1}(A)=m_H(u_{n,F}^{-1}A).$$

For $A\in\mathcal{B}(H(D_F))$, set

$$P_{N,n,F}(A)=\frac{1}{N}\#\Big\{1\le k\le N:\zeta_n(s+i\gamma_k h,F)\in A\Big\}.$$

Lemma 2. *$P_{N,n,F}$ converges weakly to the measure $V_{n,F}$ as $N\to\infty$.*

Proof. By the definitions of Q_N and $P_{N,n,F}$, we have

$$P_{N,n,F}(A)=\frac{1}{N}\#\left\{1\le k\le N:\big(p^{-i\gamma_k h}:p\in\mathbb{P}\big)\in u_{n,F}^{-1}A\right\}=Q_N(u_{n,F}^{-1}A).$$

Thus, $P_{N,n,F}=Q_N u_{n,F}^{-1}$. Therefore, the lemma is a corollary of Lemma 1, continuity of $u_{n,F}$ and Theorem 5.1 of [2]. □

The weak convergence of the measure $V_{n,F}$ as $n\to\infty$ is very important for the proof of Theorem 4. The following assertion is true.

Lemma 3. *$V_{n,F}$ converges weakly to the measure $P_{\zeta,F}$ as $n\to\infty$. Moreover, the support of $P_{\zeta,F}$ is the set*

$$S_F=\Big\{g\in H(D_F):g(s)\ne 0\ \text{ or }\ g(s)\equiv 0\Big\}.$$

Proof. The lemma is a result of [6] and [8] because $V_{n,F}$, as $n\to\infty$, and

$$\frac{1}{T}\text{meas}\big\{\tau\in[0,T]:\zeta(s+i\tau,F)\in A\big\},\quad A\in\mathcal{B}(H(D_F)),$$

as $T\to\infty$, have the same limit measure $P_{\zeta,F}$. □

To prove Theorem 4, it remains to show that the limit measure of $P_{N,F}$ as $N\to\infty$ coincides with that of $V_{n,F}$ as $n\to\infty$. For this, some mean square estimates will be applied. For convenience, we recall the Gallagher lemma which connects discrete and continuous mean squares of certain functions.

Lemma 4. *Let T_0 and $T\geqslant\delta>0$ be real numbers, and $\mathcal{T}\ne\varnothing$ be a finite set in the interval $[T_0+\frac{\delta}{2},T_0+T-\frac{\delta}{2}]$. Define*

$$N_\delta(x)=\sum_{t\in\mathcal{T},\,|t-x|<\delta}1$$

and let $S(t)$ be a complex-valued continuous function on $[T_0,T_0+T]$ having a continuous derivative on (T_0,T_0+T). Then

$$\sum_{t\in\mathcal{T}}N_\delta^{-1}(t)|S(t)|^2\le\frac{1}{\delta}\int_{T_0}^{T_0+T}|S(t)|^2\mathrm{d}t+\left(\int_{T_0}^{T_0+T}|S(t)|^2\mathrm{d}t\int_{T_0}^{T_0+T}|S'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

The proof of the lemma can be found in [15, Lemma 1.4].

Now, we recall a metric in the space $H(D_F)$. For $g_1, g_2 \in H(D_F)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D_F such that $D_F = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if K is a compact subset of D_F, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is a metric on $H(D_F)$ that induces the topology of uniform convergence on compacta.

Lemma 5. *Suppose that the estimate (1.1) is true. Then the equality*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \rho\left(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)\right) = 0$$

holds.

Proof. We start with some remarks on the mean squares of the function $\zeta(s, F)$. It is well known that, for fixed σ, $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$, the bound

$$\int_0^T |\zeta(\sigma + it, F)|^2 \mathrm{d}t \ll_\sigma T$$

is true. Hence, it follows for the same σ that, for $\tau \in \mathbb{R}$,

$$\int_0^T |\zeta(\sigma + i\tau + it, F)|^2 \mathrm{d}t \ll_\sigma T(1 + |\tau|). \tag{2.1}$$

Moreover, the Cauchy integral formula together with (2.1) leads to

$$\int_0^T |\zeta'(\sigma + i\tau + it, F)|^2 \mathrm{d}t \ll_\sigma T(1 + |\tau|). \tag{2.2}$$

Now, we apply Lemma 4. It is known that $\gamma_k \sim \frac{2\pi k}{\log k}$ as $k \to \infty$. Therefore, $\gamma_k \leqslant \frac{ck}{\log k}$ with some $c > 0$ for all $k \geqslant 2$. In Lemma 4, we take $\mathcal{T} = \{\gamma_1 h, \dots, \gamma_N h\}$, $\delta = h\left(\log \frac{N}{c \log N}\right)^{-1}$, $T_0 = \gamma_1 h - \frac{\delta}{2}$ and $T = \gamma_N h - T_0 + \frac{\delta}{2}$. Then, in view of (1.1), we find that

$$\sum_{k=1}^{N} N_\delta(\gamma_k h) = \sum_{k=1}^{N} \sum_{\substack{\gamma_l \leqslant \frac{cN}{h \log N} \\ |\gamma_k - \gamma_l| < \frac{\delta}{h}}} 1 = \sum_{\substack{0 < \gamma_l,\, \gamma_k \leqslant \frac{cN}{h \log N} \\ |\gamma_l - \gamma_k| < \frac{\delta}{h}}} \sum 1 \ll_h N.$$

Thus, applying Lemma 4 for the function $\zeta(\sigma + i\tau + i\gamma_k h, F)$, and, taking into account the estimates (2.1) and (2.2), we obtain

$$\sum_{k=1}^{N} |\zeta(\sigma + i\tau + i\gamma_k h, F)| = \sum_{k=1}^{N} \left(N_\delta(\gamma_k h) N_\delta^{-1}(\gamma_k h)\right)^{\frac{1}{2}} |\zeta(\sigma + i\tau + i\gamma_k h, F)|$$

$$
\begin{aligned}
&\le \left(\sum_{k=1}^{N} N_\delta(\gamma_k h) \sum_{k=1}^{N} N_\delta^{-1}(\gamma_k h)|\zeta(\sigma+i\tau+i\gamma_k h, F)|^2\right)^{1/2} \\
&\ll_h N^{\frac12}\Bigg(\log N \int_{\gamma_1 h-\frac{\delta}{2}}^{\gamma_N h-\gamma_1 h+\delta} |\zeta(\sigma+i\tau+it,F)|^2 \mathrm{d}t \\
&+ \left(\int_{\gamma_1 h-\frac{\delta}{2}}^{\gamma_N h-\gamma_1 h+\delta} |\zeta(\sigma+i\tau+it,F)|^2 \mathrm{d}t \int_{\gamma_1 h-\frac{\delta}{2}}^{\gamma_N h-\gamma_1 h+\delta} |\zeta'(\sigma+i\tau+it,F)|^2 \mathrm{d}t\right)^{\frac12}\Bigg)^{\frac12} \\
&\ll_h N^{\frac12}\Bigg(\log N \int_{0}^{\frac{c(h)N}{\log N}} |\zeta(\sigma+i\tau+it,F)|^2 \mathrm{d}t \\
&\quad +\left(\int_{0}^{\frac{c(h)N}{\log N}} |\zeta(\sigma+i\tau+it,F)|^2 \mathrm{d}t \int_{0}^{\frac{c(h)N}{\log N}} |\zeta'(\sigma+i\tau+it,F)|^2 \mathrm{d}t\right)^{\frac12}\Bigg)^{\frac12} \\
&\ll_h N^{\frac12}\left(\log N \frac{c(h)N}{\log N}(1+|\tau|)\right)^{\frac12} + N^{\frac12}\left(\frac{c(h)N}{\log N}(1+|\tau|)\right)^{\frac12} \\
&\ll_h N(1+|\tau|)^{\frac12} \ll_h N(1+|\tau|).
\end{aligned} \tag{2.3}
$$

Here $c(h)$ is a certain positive constant depending of h.

Let the number θ is the same as in the definition of $v_n(m)$, and

$$
l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right) n^s,
$$

where $\Gamma(s)$ denotes the Euler gamma-function. Then we have [6]

$$
\zeta_n(s,F) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,F) l_n(z)\frac{\mathrm{d}z}{z}.
$$

Hence, taking $\theta_1 > 0$, we obtain

$$
\zeta_n(s,F) - \zeta(s,F) = \frac{1}{2\pi i}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \zeta(s+z,F) l_n(z)\frac{\mathrm{d}z}{z}. \tag{2.4}
$$

We take an arbitrary fixed compact subset K of the strip D_F, denote the points of K by $s=\sigma+iv$, fix $\varepsilon>0$ such $\frac{\kappa}{2}+2\varepsilon \leqslant \sigma \leqslant \frac{\kappa+1}{2}-\varepsilon$ for $s\in K$, and choose $\theta_1 = \sigma-\varepsilon-\frac{\kappa}{2}$ and $\theta=\frac{\kappa}{2}+\varepsilon$. Then the representation (2.4) shows that, for $s\in K$,

$$
\zeta(s+i\gamma_k h,F)-\zeta_n(s+i\gamma_k h,F) \ll \int_{-\infty}^{\infty} \left|\zeta(s+i\gamma_k h-\theta_1+i\tau,F)\right| \frac{|l_n(-\theta_1+i\tau)|}{|-\theta_1+i\tau|}\mathrm{d}\tau.
$$

Hence, after a shift $\tau+v\to\tau$, we have

$$
\begin{aligned}
\zeta(s+i\gamma_k h,F)-\zeta_n(s+i\gamma_k h,F) &\ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{\kappa}{2}+\varepsilon+i(\tau+\gamma_k h),F\right)\right| \\
&\times \frac{|l_n(\frac{\kappa}{2}+\varepsilon-s+i\tau)|}{|\frac{\kappa}{2}+\varepsilon-s+i\tau|}\mathrm{d}\tau.
\end{aligned}
$$

Therefore,

$$\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+i\gamma_k h, F)-\zeta_n(s+i\gamma_k h, F)|$$
$$\ll \int_{-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=1}^{N}\left|\zeta\left(\frac{\kappa}{2}+\varepsilon+i(\tau+\gamma_k h), F\right)\right|\sup_{s\in K}\frac{|l_n(\frac{\kappa}{2}+\varepsilon-s+i\tau)|}{|\frac{\kappa}{2}+\varepsilon-s+i\tau|}\right)\mathrm{d}\tau. \quad (2.5)$$

It is well known that, uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma+i\tau) \ll \exp\{-c|\tau|\}, \quad c>0.$$

Thus, taking into account the definition of the function $l_n(s)$, we find that, for $s \in K$,

$$\frac{l_n(\frac{\kappa}{2}+\varepsilon-s+i\tau)}{\frac{\kappa}{2}+\varepsilon-s+i\tau} \ll n^{-\varepsilon}\exp\left\{-\frac{c|\tau-v|}{\theta}\right\} \ll_K n^{-\varepsilon}\exp\{-c|\tau|\}.$$

Therefore, by (2.5) and (2.3),

$$\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+i\gamma_k h, F)-\zeta_n(s+i\gamma_k h, F)|$$
$$\ll_{K,h} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|\tau|)\exp\{-c|\tau|\}\mathrm{d}\tau \ll_{K,h} n^{-\varepsilon}.$$

This shows that, for every compact set $K \subset D_F$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}\bigl|\zeta(s+i\gamma_k h, F)-\zeta_n(s+i\gamma_k h, F)\bigr| = 0,$$

and the assertion of the lemma follows from the definition of the metric ρ. □

Now, we are in position to prove Theorem 4.

Proof of Theorem 4. Let ξ_N be a random variable on a certain probability space $(\hat{\Omega}, \mathcal{A}, \mu)$ with the distribution

$$\mu\{\xi_N = \gamma_k h\} = \frac{1}{N}, \quad k=1,\dots,N.$$

Denote by $X_{n,F}$ the $H(D_F)$-valued random element with the distribution $V_{n,F}$, where $V_{n,F}$ is the limit measure in Lemma 2, and, on the probability space $(\hat{\Omega}, \mathcal{A}, \mu)$, define the $H(D_F)$-valued random element

$$X_{N,n,F} = X_{N,n,F}(s) = \zeta_n(s+i\xi_N, F).$$

Then, in view of Lemma 2,

$$X_{N,n,F} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,F}. \quad (2.6)$$

By Lemma 2, the measure $V_{n,F}$ is weakly convergent to $P_{\zeta,F}$ as $n \to \infty$. Thus,

$$X_{n,F} \xrightarrow[n\to\infty]{\mathcal{D}} P_{\zeta,F}. \tag{2.7}$$

On the above probability space, define one more $H(D_F)$-valued random element

$$Y_{N,F} = Y_{N,F}(s) = \zeta(s + i\xi_N, F).$$

Then, applying Lemma 5, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\big\{\rho(Y_{N,F}, X_{N,n,F}) \geq \varepsilon\big\}$$
$$\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N\varepsilon} \sum_{k=1}^{N} \rho\big(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)\big) = 0.$$

This equality together with (2.6) and (2.7) shows that all hypotheses of Theorem 4.2 in [2] are satisfied. Therefore, we have

$$Y_{N,F} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\zeta,F},$$

in other words, $P_{N,F}$ converges weakly to $P_{\zeta,F}$ as $N \to \infty$. The theorem is proved.

3 Proof of Theorem 3

The proof of Theorem 3 is quite standard, and is based on Theorem 4 and the Mergelyan theorem on the approximation of analytic functions by polynomials [14].

Proof of Theorem 3. By the mentioned Mergelyan theorem, there exists a polynomial $p_\varepsilon(s)$ such that

$$\sup_{s\in K} \left| f(s) - e^{p_\varepsilon(s)} \right| < \frac{\varepsilon}{2}. \tag{3.1}$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D_F) : \sup_{s\in K} |g(s) - e^{p_\varepsilon(s)}| < \frac{\varepsilon}{2} \right\}.$$

Clearly, $e^{p_\varepsilon(s)} \in S$. Therefore, in virtue of Lemma 3, the set G_ε is an open neighbourhood of an element of the support of the measure $P_{\zeta,F}$. Hence, by a property of the support,

$$P_{\zeta,F}(G_\varepsilon) > 0, \tag{3.2}$$

and Theorem 4 together with the equivalent of weak convergence of probability measures in terms of open sets [2, Theorem 2.1] implies

$$\liminf_{N\to\infty} P_{N,F}(G_\varepsilon) \geq P_{\zeta,F}(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,F}$ and G_ε, and (3.1) prove the first assertion of the theorem.

For the proof of the second assertion of the theorem, define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the boundary $\partial \hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\},$$

therefore, $\partial G_{\varepsilon_1} \bigcap \partial G_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. This remark implies that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,F}$, i.e., $P_{\zeta,F}(\partial \hat{G}_\varepsilon) = 0$, for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 4 together with the equivalent of weak convergence of probability measures in terms of continuity sets [2, Theorem 2.1] gives the equality

$$\lim_{N \to \infty} P_{N,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \tag{3.3}$$

for all but at most countably many $\varepsilon > 0$. The definitions of the sets G_ε and $\hat{G}_\varepsilon$, and inequality (3.1) imply the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Hence, in view of (3.2), we have $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$. The latter inequality, the definitions of $P_{N,F}$ and $\hat{G}_\varepsilon$, and (3.3) prove the second assertion of the theorem. The theorem is proved.

References

[1] A. Balčiūnas, V. Garbaliauskienė, J. Karaliūnaitė, R. Macaitienė, J. Petuškinaitė and A. Rimkevičienė. Joint discrete approximation of a pair of analytic functions by periodic zeta-functions. *Math. Modell. Analysis*, **25**(1):71–87, 2020. https://doi.org/10.3846/mma.2020.10450.

[2] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.

[3] R. Garunkštis and A. Laurinčikas. Discrete mean square of the Riemann zeta-function over imaginary parts of its zeros. *Periodica Math. Hung.*, **76**(2):217–228, 2018. https://doi.org/10.1007/s10998-017-0228-6.

[4] R. Garunkštis and A. Laurinčikas. The Riemann hypothesis and universality of the Riemann zeta-function. *Math. Slovaca*, **68**(4):741–748, 2018. https://doi.org/10.1515/ms-2017-0141.

[5] R. Garunkštis, A. Laurinčikas and R. Macaitienė. Zeros of the Riemann zeta-function and its universality. *Acta Arith.*, **181**(2):127–142, 2017. https://doi.org/10.4064/aa8583-5-2017.

[6] A. Kačėnas and A. Laurinčikas. On Dirichlet series related to certain cusp forms. *Lith. Math. J.*, **38**:64–76, 1998. https://doi.org/10.1007/BF02465545.

[7] A. Laurinčikas. Non-trivial zeros of the Riemann zeta-function and joint universality theorems. *J. Math. Anal. Appl.*, **475**(1):395–402, 2019. https://doi.org/10.1016/j.jmaa.2019.02.047.

[8] A. Laurinčikas and K. Matsumoto. The universality of zeta-functions attached to certain cusp forms. *Acta Arith.*, **98**:345–359, 2001. https://doi.org/10.4064/aa98-4-2.

[9] A. Laurinčikas, K. Matsumoto and J. Steuding. Discrete universality of L-functions for new forms. *Math. Notes*, **78**:551–558, 2003. https://doi.org//10.1007/s11006-005-0153-5.

[10] A. Laurinčikas, K. Matsumoto and J. Steuding. Discrete universality of L-functions for new forms. II. *Lith. Math. J.*, **56**:207–218, 2016. https://doi.org/10.1007/s10986-016-9314-3.

[11] A. Laurinčikas and J. Petuškinaitė. Universality of Dirichlet L-functions and non-trivial zeros of the Riemann zeta-function. *Sb. Math.*, **210**(12):1753–1773, 2019. https://doi.org/10.1070/SM9194.

[12] A. Laurinčikas, D. Šiaučiūnas and A. Vaiginytė. Extension of the discrete universality theorem for zeta-functions of certain cusp forms. *Nonlinear Analysis: Modell. Control*, **23**(6):961–973, 2018. https://doi.org/10.15388/NA.2018.6.10.

[13] A. Laurinčikas, D. Šiaučiūnas and A. Vaiginytė. On joint approximation of analytic functions by non-linear shifts of zeta-functions of certain cusp forms. *Nonlinear Analysis: Modell. Control*, **25**(1):108–125, 2020. https://doi.org/10.15388/namc.2020.25.15734.

[14] S.N. Mergelyan. Uniform approximations to functions of complex variable. *Usp. Mat. Nauk.*, **7**(2):31–122, 1952 (in Russian).

[15] H.L. Montgomery. *Topics in Multiplicative Number Theory*. Lect. Notes Math., Vol. 227, Springer, Berlin, Heidelberg, New York, 1971. https://doi.org/10.1007/BFb0060851.

[16] H.L. Montgomery. The pair correlation of zeros of the zeta function. In H.G. Diamond(Ed.), *Analytic Number Theory*, volume 24 of *Proc. Sympos. Pure Math.*, pp. 181–193. Amer. Math. Soc., Providence, RI, 1973. https://doi.org/10.1090/pspum/024/9944.