Mathematical Modelling and Analysis
Volume 26, Issue 1, 21–33, 2021
https://doi.org/10.3846/mma.2021.12445

A Weighted Version of the Mishou Theorem

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Received April 9, 2020; revised September 27, 2020; accepted September 27, 2020

Abstract. In 2007, H. Mishou obtained a joint universality theorem for the Riemann and Hurwitz zeta-functions $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental parameter α on the approximation of a pair of analytic functions by shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$, $\tau \in \mathbb{R}$. In the paper, the Mishou theorem is generalized for the set of above shifts having a weighted positive lower density. Also, the case of a positive density is considered.

Keywords: Hurwitz zeta-function, Mishou theorem, Riemann zeta-function, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, and the Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter $0 < \alpha \leqslant 1$ are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \quad \text{and} \quad \zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and have analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The functions $\zeta(s)$ and $\zeta(s,\alpha)$ play an important role not only in analytic number theory but in mathematics in general. The definitions of $\zeta(s)$ and $\zeta(s,\alpha)$ are similar, however, their analytic properties are quite different. For example, since the function $\zeta(s)$, for $\sigma > 1$, has the Euler product over primes

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

$\zeta(s) \neq 0$ in the half-plane $\sigma > 1$, while the function $\zeta(s,\alpha)$ has zeros in that half plane if $\alpha \neq 1$ or $1/2$. On the other hand, the functions $\zeta(s)$ and $\zeta(s,\alpha)$ have a common feature, they are universal in the sense that their shifts $\zeta(s+i\tau)$ and $\zeta(s+i\tau,\alpha)$ approximate wide classes of analytic functions. We recall that universality of the function $\zeta(s)$ was discovered by S.M. Voronin in [31]. For a statement of the Voronin theorem, it is convenient to use the following notation. For $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K, and by $H_0(K)$ the subclass of $H(K)$ of non-vanishing functions on K. Then the modern version of the Voronin theorem, see, for example, [1, 6, 13, 30] asserts that, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The latter inequality shows that there are infinitely many shifts $\zeta(s+i\tau)$ approximating a given function $f(s) \in H_0(K)$. Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Universality of the Hurwitz zeta-function is a more complicated problem. At the moment, the following result is known. Suppose that α is a transcendental or rational $\neq 1, 1/2$. Then, for every $K \in \mathcal{K}$, $f(s) \in H(K)$, and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

The case of rational α was obtained by Voronin [32] and B. Bagchi [1], while the case of transcendental α was treated by S.M. Gonek [6], and, by a different method, in [23]. In [14], the transcendence of α was replaced by a weaker condition on the linear independence of the set $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ over the field of rational numbers $\mathbb{Q}$.

H. Mishou in [29] began to study a joint approximation property of the functions $\zeta(s)$ and $\zeta(s,\alpha)$. More precisely, he proved that if α is transcendental, then, for every $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$, $f_2 \in H(K_2)$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\Big\} > 0.$$

The Mishou theorem is the first so-called mixed joint universality theorem because the function $\zeta(s)$ has Euler's product over primes, while the function

$\zeta(s,\alpha)$ with transcendental α has no such a product. Mixed joint universality theorems were studied in [2,5,7,8,9,10,11,15,16,17,18,19,20,21,22,24].

The aim of this paper, is a joint weighted universality theorem for the functions $\zeta(s)$ and $\zeta(s,\alpha)$. The weighted universality of zeta-functions was began to study in [12]. In weighted universality theorems, the positivity of a lower density of the shifts approximating a given analytic function is replaced by the positivity of that weighted analogue. Let $w(\tau)$ be positive function for $\tau \geqslant T_0 > 0$ such that

$$\lim_{T\to\infty} W(T,w) = \infty, \quad W(T,w) = \int_{T_0}^{T} w(\tau)\,\mathrm{d}\tau,$$

and, for every interval $[a,b] \subset [T_0,\infty)$, the variation $\mathrm{V}_a^b\, w$ satisfies the inequality $\mathrm{V}_a^b\, w \leqslant cw(a)$ with certain $c > 0$. Moreover, let $I(A)$ denote the indicator function of the set A. Under the above hypotheses on the weight function w, it was obtained in [12] that, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau\in[T_0,T] : \sup_{s\in K} |\zeta(s+i\tau)-f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

A weighted discrete universality for $\zeta(s)$ was proved in [25]. Weighted universality theorems for periodic zeta-functions were obtained in [26,27].

A weighted universality theorem for the Hurwitz zeta-function was proved in [3]. Denote by W the above class of weight functions.

Theorem 1. *Suppose that α is transcendental and $w \in W$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau\in[T_0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

The main result of this paper is the following weighted theorem.

Theorem 2. *Suppose that α is transcendental and $w \in W$. Let $K_1, K_2 \in \mathcal{K}$ and $f(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f(s)| < \varepsilon, \right.\right.$$
$$\left.\left. \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f(s)| < \varepsilon, \right.\right.$$
$$\left.\left. \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0$$

exists for all but at most countably many $\varepsilon > 0$.

If $w(\tau) = 1$, then the first assertion of Theorem 2 reduces to the Mishou theorem [29]. For example, we may take $w(\tau) = 1/\tau$ and $\alpha = 1/\mathrm{e}$.

For the proof of Theorem 2, we will use the probabilistic approach based on weak convergence of probability measures in the space of analytic functions.

2 A weighted limit theorem on the product of two tori

In what follows, we denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, by $\mathbb{P}$ the set of all prime numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define two tori $\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p$ and $\Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m$, where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With product topology and pointwise multiplication, the infinite-dimensional tori Ω_1 and Ω_2 are compact topological Abelian groups. Therefore, $\Omega = \Omega_1 \times \Omega_2$ is again a compact topological Abelian group. Hence, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega_1(p)$ the pth component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and by $\omega_2(m)$ the mth component of an element $\omega_2 \in \Omega_2$, $m \in \mathbb{N}_0$. The elements of Ω are of the form $\omega = (\omega_1, \omega_2)$.

In this section, we will consider the weak convergence for

$$Q_{T,w}(A) = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \left(\left(p^{-i\tau} : p \in \mathbb{P}\right), \left((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\right)\right) \in A\right\}\right) \mathrm{d}\tau, \quad A \in \mathcal{B}(\Omega).$$

Theorem 3. *Suppose that α is transcendental and $w \in W$. Then $Q_{T,w}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. The characters of the group Ω are of the form

$$\prod_{p\in\mathbb{P}}{}' \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}' \omega_2^{l_m}(m),$$

where the sign "$'$" means that only a finite number of integers k_p and l_m are distinct from zero. Therefore, the Fourier transform $g_{T,w}(\underline{k}, \underline{l})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, $\underline{l} = (k_p : l_m \in \mathbb{Z}, m \in \mathbb{N}_0)$, of $Q_{T,w}$ is defined by

$$g_{T,w}(\underline{k}, \underline{l}) = \int_{\Omega} \prod_{p\in\mathbb{P}}{}' \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}' \omega_2^{l_m}(m) \,\mathrm{d}Q_{T,w}.$$

Therefore, by the definition of $Q_{T,w}$,

$$\begin{aligned} g_{T,w}(\underline{k}, \underline{l}) &= \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \prod_{p\in\mathbb{P}}{}' p^{-ik_p\tau} \prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{-il_m\tau} \,\mathrm{d}\tau \\ &= \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \exp\left\{ -i\tau\left(\sum_{p\in\mathbb{P}}{}' k_p \log p + \sum_{m\in\mathbb{N}_0}{}' l_m \log(m+\alpha)\right)\right\} \mathrm{d}\tau. \end{aligned} \tag{2.1}$$

Clearly,

$$g_{T,w}(\underline{0}, \underline{0}) = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \,\mathrm{d}\tau = 1. \tag{2.2}$$

Suppose that $(\underline{k},\underline{l}) \neq (\underline{0},\underline{0})$. Then

$$A(\underline{k},\underline{l}) \stackrel{def}{=} \sum_{p\in\mathbb{P}}{}' k_p \log p + \sum_{m\in\mathbb{N}_0}{}' l_m \log(m+\alpha) \neq 0. \tag{2.3}$$

Actually, if the latter inequality is not true, then

$$\prod_{p\in\mathbb{P}}{}' p^{k_p} \prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{l_m} = 1.$$

From this, it follows that

$$\prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{l_m}$$

is a rational number. However, this contradicts the transcendence of α. If all $l_m = 0$, then $\sum'_{p\in\mathbb{P}} k_p \log p \neq 0$ because the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers. Thus, (2.3) is true. Now, by (2.1), we find

$$g_{T,w}(\underline{k},\underline{l}) = \frac{1}{-iW(T,w)A(\underline{k},\underline{l})} \int_{T_0}^{T} w(\tau)\,\mathrm{d}\exp\{-i\tau A(\underline{k},\underline{l})\}$$
$$\ll (W(T,w)|A(\underline{k},\underline{l})|)^{-1}\Big(1 + \int_{T_0}^{T} |\mathrm{d}w(\tau)|\Big) \ll (W(T,w)|A(\underline{k},\underline{l})|)^{-1}$$

in view of a property of the variation of $w(\tau)$. Since $\lim_{T\to\infty} W(T,w) = \infty$, this shows that

$$\lim_{T\to\infty} g_{T,w}(\underline{k},\underline{l}) = 0.$$

Therefore, by (2.2),

$$\lim_{T\to\infty} g_{T,w}(\underline{k},\underline{l}) = \begin{cases} 1 & \text{if } (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ 0 & \text{if } (\underline{k},\underline{l}) \neq (\underline{0},\underline{0}), \end{cases}$$

and the theorem is proved because the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H. □

3 Case of absolute convergence

Theorem 3 implies a weighted joint limit theorem in the space $H^2(D)$, where $H(D)$ is the space of analytic functions on D endowed with the topology of uniform convergence on compacta. Thus, let $\theta > 1/2$ be a fixed number, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\Big\{ -\Big(\frac{m}{n}\Big)^{\theta} \Big\},$$

and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m,\alpha) = \exp\Big\{ -\Big(\frac{m+\alpha}{n+\alpha}\Big)^{\theta} \Big\}.$$

Define the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then the latter series are absolutely convergent for $\sigma > 1/2$, see [13, 23], respectively. For brevity, let

$$\underline{\zeta}_n(s,\alpha) = \left(\zeta_n(s), \zeta_n(s,\alpha)\right).$$

Extend the functions $\omega_1(p)$, to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega_1^l(p), \quad m \in \mathbb{N},$$

and, additionally to $\zeta_n(s)$ and $\zeta_n(s,\alpha)$, define

$$\zeta_n(s,\omega_1) = \sum_{m=1}^{\infty} \frac{\omega_1(m) v_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s,\omega_2,\alpha) = \sum_{m=0}^{\infty} \frac{\omega_2(m) v_n(m,\alpha)}{(m+\alpha)^s},$$

and put $\underline{\zeta}_n(s,\omega,\alpha) = \left(\zeta_n(s,\omega_1), \zeta_n(s,\omega_2,\alpha)\right)$. Obviously, the series $\zeta_n(s,\omega_1)$ and $\zeta_n(s,\omega_2,\alpha)$ are absolutely convergent for $\sigma > 1/2$ as well.

Consider the function $u_n : \Omega \to H^2(D)$ given by $u_n(\omega) = \underline{\zeta}_n(s,\omega,\alpha)$. Since the above seeries are absolutely convergent for $\sigma > 1/2$, the function $u_n(\omega)$ is continuous. For $A \in \mathcal{B}(H^2(D))$, define

$$P_{T,n,w}(A) = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0,T] : \underline{\zeta}_n(s+i\tau,\alpha) \in A\right\}\right) \mathrm{d}\tau.$$

Then we have $P_{T,n,w}(A) = Q_{T,w}(u^{-1}A)$. Thus, the equality $P_{T,n,w} = Q_{T,w}u^{-1}$ is true. This, the continuity of u_n, Theorem 3 together with Theorem 5.1 of [4] lead to the following theorem.

Theorem 4. *Suppose that α is transcendental and $w \in W$. Then $P_{T,n,w}$ converges weakly to the measure $V_n \stackrel{def}{=} m_H u_n^{-1}$ as $T \to \infty$.*

The measure V_n plays an important role in the proof of the limit theorem for

$$P_{T,w}(A) = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0,T] : \underline{\zeta}(s+i\tau,\alpha) \in A\right\}\right) \mathrm{d}\tau,$$
$$A \in \mathcal{B}(H^2(D)),$$

where $\underline{\zeta}(s,\alpha) = (\zeta(s), \zeta(s,\alpha))$. From the proof of the Mishou theorem [29], the following properties of V_n follows. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element

$$\underline{\zeta}(s,\omega,\alpha) = \left(\prod_{p\in\mathbb{P}} \left(1 - \frac{\omega_1(p)}{p^s}\right)^{-1}, \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s}\right),$$

and let $P_{\underline{\zeta}}$ be the distribution of $\underline{\zeta}(s,\omega,\alpha)$, i. e.,

$$P_{\underline{\zeta}}(A) = m_H \left\{\omega \in \Omega : \underline{\zeta}(s,\omega,\alpha) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)).$$

Moreover, let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Under the above notation, we have

Lemma 1. *Suppose that α is transcendental. Then V_n converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$. Moreover, the support of $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

To prove that $P_{T,w}$, as $T \to \infty$, also converges weakly to the measure $P_{\underline{\zeta}}$, some approximation of $\underline{\zeta}(s,\alpha)$ by $\underline{\zeta}_n(s,\alpha)$ is needed.

4 Approximation in the mean

For $g_1, g_2 \in H(D)$, define

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s)-g_2(s)|}{1+\sup_{s\in K_l} |g_1(s)-g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact subsets such that $D = \bigcup\limits_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then K lies in some K_l. Then ρ is a metric on $H(D)$ that induces its topology of uniform convergence on compacta.

Now, let $\underline{g}_1 = (g_{11}, g_{12})$, $\underline{g}_2 = (g_{21}, g_{22}) \in H^2(D)$. Then putting

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant 2} \rho(g_{1j}, g_{2j})$$

gives a metric on $H^2(D)$ inducing the product topology.

The following statement is true.

Theorem 5. *Suppose that $w \in W$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau)\underline{\rho}\left(\underline{\zeta}(s+i\tau,\alpha), \underline{\zeta}_n(s+i\tau,\alpha)\right) \mathrm{d}\tau = 0$$

for all $0 < \alpha \leqslant 1$.

Proof. By the definition of the metric ρ, it suffices to prove the equalities

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau)\rho\left(\zeta(s+i\tau), \zeta_n(s+i\tau)\right) \mathrm{d}\tau = 0 \tag{4.1}$$

and

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau)\rho\left(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)\right) \mathrm{d}\tau = 0. \tag{4.2}$$

Obviously, (4.1) is a corollary of (4.2) with $\alpha = 1$. Moreover, to prove (4.2) it suffices to show that, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \, d\tau = 0. \quad (4.3)$$

Let

$$l_n(s,\alpha) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right)(n+\alpha)^s, \quad n \in \mathbb{N},$$

where $\Gamma(s)$ is the Euler gamma-function. Then the classical Mellin formula implies, for $\sigma > 1/2$, the equality

$$\zeta_n(s,\alpha) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha) \frac{dz}{z}. \quad (4.4)$$

We take an arbitrary compact set $K \subset D$, and fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for points $s = \sigma + iv \in K$. Then, by (4.4) and the residue theorem, for $\theta_1 > 0$,

$$\zeta_n(s,\alpha) - \zeta(s,\alpha) = \frac{1}{2\pi i} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha) \frac{dz}{z} + R_n(s,\alpha), \quad (4.5)$$

where $R_n(s,\alpha) = l_n(1-s,\alpha)/(1-s)$. Suppose that $\theta_1 = \sigma - \varepsilon - 1/2$. Then (4.5) shows that, for $s \in K$,

$$|\zeta_n(s,\alpha) - \zeta(s,\alpha)| \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |\zeta(s+i\tau-\theta_1+it,\alpha)| \frac{|l_n(-\theta_1+it,\alpha)|}{|-\theta_1+it|} dt + |R_n(s+i\tau,\alpha)|.$$

Hence, after shifting $v+t$ to t, we obtain

$$\frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \, d\tau \ll I_1 + I_2, \quad (4.6)$$

where

$$I_1 = \int_{-\infty}^{\infty} w(\tau) \left(\frac{1}{W(T,w)} \int_{T_0}^{T} |\zeta(1/2+\varepsilon+i(t+\tau),\alpha)| \, d\tau \right) \times \sup_{s\in K} \frac{|l_n(1/2+\varepsilon-s+it,\alpha)|}{|1/2+\varepsilon-s+it|} dt,$$

$$I_2 = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |R_n(s+i\tau,\alpha)| \, d\tau.$$

It is well known that $\Gamma(\sigma+it) \ll \exp\{-c|t|\}$ uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ for every $\sigma_1 < \sigma_2$ with an absolute constant $c > 0$. Therefore, putting $\theta = 1/2+\varepsilon$, we find that, for $s \in K$,

$$\frac{|l_n(1/2+\varepsilon-s+it,\alpha)|}{|1/2+\varepsilon-s+it|} = \frac{(n+\alpha)^{1/2+\varepsilon-\sigma}}{\theta} \left| \Gamma\Big(\frac{1/2+\varepsilon-\sigma}{\theta} + \frac{i(t-v)}{\theta}\Big) \right| \ll_\alpha \frac{n^{-\varepsilon}}{\theta} \exp\left\{ -c\frac{|t-v|}{\theta} \right\} \ll_{K,\alpha} n^{-\varepsilon} \exp\{-c_1|t|\} \quad (4.7)$$

with $c_1 > 0$. In [3] it was obtained that, for σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,

$$\int_{T_0}^{T} w(\tau)\left|\zeta(\sigma+i(t+\tau),\alpha)\right|^2 \mathrm{d}t \ll W(t,w)(1+|t|^2).$$

Hence,

$$\begin{aligned}&\int_{T_0}^{T} w(\tau)\left|\zeta(\sigma+i(t+\tau),\alpha)\right|^2 \mathrm{d}\tau\\ &\ll \Big(\int_{T_0}^{T} w(\tau)\,\mathrm{d}\tau \int_{T_0}^{T} w(\tau)\left|\zeta(1/2+\varepsilon+i(t+\tau),\alpha)\right|^2 \mathrm{d}\tau\Big)^{1/2} \ll W(t,w)(1+|t|^2).\end{aligned}$$

This together with (4.7) shows that

$$I_1 \ll_K n^{-\varepsilon} \int_{-\infty}^{\infty} (1+|t|)\exp\{-c_1|t|\}\,\mathrm{d}t \ll_{K,\alpha} n^{-\varepsilon}. \tag{4.8}$$

Similarly, we find that, for $s \in K$,

$$|R_n(s+i\tau,\alpha)| \ll_\alpha n^{1-\sigma}\exp\Big\{-c\frac{|\tau-v|}{\theta}\Big\} \ll_{K,\alpha} n^{1-\sigma}\exp\{-c_2|\tau|\}$$

with $c_2 > 0$. Thus,

$$I_2 \ll_{K,\alpha} n^{1/2-2\varepsilon}\frac{1}{W(T,w)}\int_{T_0}^{T} w(\tau)\exp\{-c_2|\tau|\}\,\mathrm{d}\tau \ll_{K,\alpha} \frac{n^{1/2-2\varepsilon}}{W(T,w)}.$$

If $T \to \infty$, then $I_2 \to 0$, because $W(T,w) \to \infty$. Moreover, by (4.8), if $n \to \infty$, then $I_1 \to 0$. Therefore, (4.6) implies (4.3). The lemma is proved. □

5 A limit theorem for $\underline{\zeta}(s,\alpha)$

Now we are ready to prove the weak convergence for $P_{T,w}$ as $T \to \infty$.

Theorem 6. *Suppose that α is transcendental and $w \in W$. Then $P_{T,w}$ converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof. On a certain probability space with measure μ, define a random variable $\theta_{T,w}$ by

$$\mu\{\theta_{T,w} \in A\} = \frac{1}{W(T,w)}\int_{T_0}^{T} w(\tau)I(A)\,\mathrm{d}\tau, \quad A \in \mathcal{B}(\mathbb{R}).$$

Consider the $H^2(D)$-valued random element

$$\underline{X}_{T,n,w} = \underline{X}_{T,n,w}(s) = \underline{\zeta}_n(s+i\theta_{T,w},\alpha).$$

Then, in view of Theorem 4,

$$\underline{X}_{T,n,w} \xrightarrow[T\to\infty]{\mathcal{D}} Y_n, \tag{5.1}$$

where Y_n is the $H^2(D)$-valued random element with the distribution V_n. Lemma 1 implies the relation

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}}.$$

Moreover, an application of Theorem 5 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left(\underline{\rho}\left(\underline{X}_{T,w}(s), \underline{X}_{T,n,w}(s)\right) \geqslant \varepsilon\right) \\ \ll \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon W(T,w)} \int_{T_0}^{T} w(\tau)\underline{\rho}\left(\underline{\zeta}(s+i\tau,\alpha), \underline{\zeta}_n(s+i\tau,\alpha)\right) \mathrm{d}\tau=0, \quad (5.2)\end{aligned}$$

where the $H^2(D)$-valued random element $\underline{X}_{T,w} = \underline{X}_{T,w}(s)$ is defined by

$$\underline{X}_{T,w}(s) = \underline{\zeta}(s + i\theta_{T,w}, \alpha).$$

Now, relations (5.1)–(5.2) show that all hypotheses of Theorem 4.2 from [4] are satisfied. Therefore, we obtain that

$$\underline{X}_{T,w} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\zeta}},$$

and this is equivalent to the assertion of the theorem. □

6 Proof of universality

Theorem 2 follows easily from Theorem 6 and the Mergelyan theorem on the approximation of analytic functions by polynomials [28].

Proof. (Proof of Theorem 2). By the Mergelyan theorem, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} \left|f_1(s) - \mathrm{e}^{p_1(s)}\right| < \frac{\varepsilon}{2}, \quad \sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{6.1}$$

Define the set

$$G_\varepsilon = \left\{g_1, g_2 \in H(D) : \sup_{s\in K_1} \left|g_1(s) - \mathrm{e}^{p_1(s)}\right| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2}\right\}.$$

We observe that, in virtue of Lemma 1, $(\mathrm{e}^{p_1(s)}, p_2(s))$ is an element of the support of the measure $P_{\underline{\zeta}}$. Since G_ε is an open neighbourhood of an element of the support of $P_{\underline{\zeta}}$, the inequality

$$P_{\underline{\zeta}}(G_\varepsilon) > 0 \tag{6.2}$$

is true. Therefore, using the equivalent of the weak convergence of probability measures in terms of open sets and taking into account Theorem 6, we have

$$\liminf_{T\to\infty} P_{T,w}(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

Hence, by the definitions of $P_{T,w}$ and G_ε,

$$\liminf_{T\to\infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I\Big(\Big\{\tau \in [T_0,T] : \sup_{s\in K_1} \Big|\zeta(s+i\tau) - \mathrm{e}^{p_1(s)}\Big| < \frac{\varepsilon}{2}, \\ \sup_{s\in K_2} \big|\zeta(s+i\tau,\alpha) - p_2(s)\big| < \frac{\varepsilon}{2}\Big\}\Big)\,\mathrm{d}\tau > 0. \tag{6.3}$$

It remains to replace $\mathrm{e}^{p_1(s)}$ and $p_2(s)$ by $f_1(s)$ and $f_2(s)$, respectively. Suppose that τ satisfy inequalities

$$\sup_{s\in K_1} \Big|\zeta(s+i\tau) - \mathrm{e}^{p_1(s)}\Big| < \frac{\varepsilon}{2}, \quad \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - p_2(s)| < \frac{\varepsilon}{2}.$$

Then inequalities (6.1) imply

$$\sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \quad \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon.$$

Consequently,

$$\Big\{\tau \in [T_0,T] : \sup_{s\in K_1} \Big|\zeta(s+i\tau) - \mathrm{e}^{p_1(s)}\Big| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - p_2(s)| < \frac{\varepsilon}{2}\Big\} \\ \subset \Big\{\tau \in [T_0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\}.$$

This and (6.3) prove the first assertion of the theorem.

Define one more set

$$\hat{G}_\varepsilon = \Big\{g_1, g_2 \in H(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon\Big\}.$$

Then the boundaries $\partial\hat{G}_{\varepsilon_1}$ and $\partial\hat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. This shows that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, using the equivalent of weak convergence of probability measures in terms of continuity sets, we obtain by Theorem 6 that

$$\lim_{T\to\infty} P_{T,w}(\hat{G}_\varepsilon) = P_{\underline{\zeta}}(\hat{G}_\varepsilon) \tag{6.4}$$

for all but at most countably many $\varepsilon > 0$. Moreover, inequalities (6.1) imply the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Thus, by (6.2), the inequality $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$ holds. This, the definitions of $P_{T,w}$ and $\hat{G}_\varepsilon$, and (6.4) prove the second assertion of the theorem. □

Acknowledgements

The research of the first author is funded by the European Social Fund (project No. 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMT LT).

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