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1. Introduction

In the elasticity and plasticity theories the extremum energy principles allow to create variational formulations of problems and various schemes of finite elements. The energy principles in the theory of plasticity were created and improved by such famous scientists as G. Colonnetti, H. Greenberg, A. Čyras, A. Borkowski, D. Drucker, W. Prager W. Koiter, E. Melan, A. Gvozdev etc. Some of them are even called by the names of these scientists. A. Čyras has formulated the energy principles [1-5], defining a limit state of elastic-plastic discrete structures and deformable body, and created their mathematical models, determined dual relations. He has also developed the static and kinematic theorems of analysis and optimization of the elastic-plastic body under monotonically increasing (simple) and repeated-variable loading. The energy principles of the mechanics of the elastic-plastic body formulated by foreign scientists are published in a lot of articles and monographs [6-10 etc]. In the monograph [11] the extremum principles of the dynamics of the perfectly plastic body are investigated.

Though most extremum principles were formulated independently, but later some relations of them were observed. First of all, it was the duality of static and kinematic formulations, the equivalence of some energy principles. For example, W. Prager and H. Symonds [6], A. Procenko [10] have proposed the principles of residual stresses equivalent for Haar-Karman's principle. Some principles, for example, those of Colonnetti, Greenberg, Haar-Karman, turned out to be separate cases of other general principles [9]. The aims of this work are: a) to carry out analysis of extremum energy principles of the elastic-plastic body under monotonically increasing loading and to illustrate new relations in one article; b) to show that most energy principles in the elasticity and plasticity theory, even principles defining the limit state of a perfectly plastic body, can be received from elastic-plastic body principle of a minimum of total complementary energy or from dual to it the principle of a minimum of total strain energy.

2. The principles of a minimum of total complementary energy

Let's assume that the elastic-plastic body is effected by an external load $\mathbf{F(x)}$, initial known strains $\varepsilon_0(\mathbf{x})$ and initial displacements (support settlements) $\mathbf{u}_0(\mathbf{x})$. Suppose that at its surface S_f the intensity and direction of the load $\mathbf{F(x)}$ are known, but the displacements $\mathbf{u}(\mathbf{x})$ are unknown. In other parts of the surface S_u displacements $\mathbf{u}_0(\mathbf{x})$ are given (fixed), while the forces (reactions) $\mathbf{F}_r(\mathbf{x})$ are unknown. The body surface $S = S_f \cup S_u$. In order to abridge and simplify the expressions of equations and functionals, further the dependence of vectors-functions upon the coordinates $\mathbf{x}$ is not indicated, i.e. the markings $\sigma(\mathbf{x}) = \sigma$, $\mathbf{u}(\mathbf{x}) = \mathbf{u}$ etc. are accepted.

The functional of total complementary energy

$$U_1 = \frac{1}{2}\int_V \sigma^T [d]\sigma\, dV + \int_V \sigma^T \varepsilon_0\, dV + \frac{1}{2}\int_V \lambda^T [H]\lambda\, dV - \int_{S_u} \mathbf{u}_0^T [A_s]\sigma\, dS, \tag{1}$$

where $[d]$ is a flexibility matrix of body elementary element; $[H]$ is the strengthening matrix; λ is the vector of plastic multipliers. The third member of this functional expresses the complementary energy of strengthening. A statically admissible stresses vector-function σ is defined by the equilibrium equations

$$[\mathscr{A}]\sigma = \mathbf{0} \quad \in V, \qquad [A_s]\sigma = \mathbf{F} \quad \in S_f \tag{2}$$

and the yield condition

$$\mathbf{f}_0(C) - \mathbf{f}(\sigma) + [H]\lambda \geq \mathbf{0} \quad \in V, \tag{3}$$

where C is function of material plasticity constant, $\lambda \geq \mathbf{0}$. At the investigation moment T the vector of plastic multipliers

$$\lambda(\mathbf{x}) = \int_0^T \dot{\lambda}(\mathbf{x},t)\,dt,$$

where $\dot{\lambda}(\mathbf{x},t)$ is the vector of plastic multipliers velocities.

The principle of a minimum of total complementary energy is formulated in this way:

The actual vector of all statically admissible vectors of stresses in the elastic-plastic body is the vector for which the total complementary energy is a minimum.

This principle is expressed by the following extremum problem:

$$\frac{1}{2}\int_V \sigma^T [d]\sigma\, dV + \int_V \sigma^T \varepsilon_0\, dV + \frac{1}{2}\int_V \lambda^T [H]\lambda\, dV - $$
$$- \int_{S_u} \mathbf{u}_0^T [A_s]\sigma\, dS \Rightarrow \mathbf{min}$$

under the conditions (4)

$$\mathbf{f}_0(C) - \mathbf{f}(\sigma) + [H]\lambda \geq \mathbf{0}, \quad \lambda \geq \mathbf{0} \quad \in V;$$
$$-[\mathscr{A}]\sigma = \mathbf{0} \quad \in V, \qquad [A_s]\sigma = \mathbf{F} \quad \in S_f.$$

This is the static formulation of stress-strain field analysis problem of a linear strengthening body. This formulation allows to determine the distribution of stresses.

From this extremum principle, the whole line of other energy principles can be deduced. Taking $[H] = [0]$, the principle of a minimum of total complementary energy for perfectly elastic-plastic body is:

$$\frac{1}{2}\int_V \sigma^T [d]\sigma\, dV + \int_V \sigma^T \varepsilon_0\, dV - \int_{S_u} \mathbf{u}_0^T [A_s]\sigma\, dS \Rightarrow \mathbf{min}$$

under the conditions (5)

$$\mathbf{f}_0(C) - \mathbf{f}(\sigma) \geq \mathbf{0} \quad \in V,$$
$$-[\mathscr{A}]\sigma = \mathbf{0} \quad \in V, \qquad [A_s]\sigma = \mathbf{F} \quad \in S_f.$$

When initial deformations do not exist ($\varepsilon_0 = \mathbf{0}$), the problem of mathematical programming (5) expresses the Haar-Karman's principle [7,12]. If initial deformations ε_0 are identified with given plastic deformations ε_p, we will receive Colonnetti's principle [7,13]:

When the plastic deformations are known, the actual vector of all statically admissible vectors of stresses in the elastic-plastic body is the one for which the total complementary energy is a minimum.

In case of a linear elastic body, when elastic deformations are not limited and the plastic constant is non-limited large ($C = \infty$), the problem (5) expresses the classical Castigliano's principle [14,15]. At last, when only an external load effects the elastic body ($\varepsilon_0 = \mathbf{0}$ and $\mathbf{u}_0 = \mathbf{0}$), the principle of a minimum of total complementary energy formulated by us coincides with Menabrea theorem [9] (a separate case of Castigliano's principle).

The extremum principle (4) can be expressed by residual stresses σ_r. Using the relationships

$$\sigma = \sigma_e + \sigma_r,$$
$$[d]\sigma_e + \varepsilon_0 - [\mathscr{A}]^T \mathbf{u}_e = \mathbf{0} \quad \in V,$$
$$\mathbf{u}_e = \mathbf{u}_0 \quad \in S_u$$

and

$$\int_V \sigma_r^T [d]\sigma_e dV = \int_V \sigma_r^T [\mathscr{A}]^T \mathbf{u}_e dV - \int_V \sigma_r^T \varepsilon_0 dV =$$
$$= -\int_V \mathbf{u}_e^T [\mathscr{A}]\sigma_r dV + \int_{S_f} \mathbf{u}_e^T [A_s]\sigma_r dS +$$
$$+ \int_{S_u} \mathbf{u}_e^T [A_s]\sigma_r dS - \int_V \sigma_r^T \varepsilon_0 dV,$$

the second expression of the functional of a total complementary energy is received:

$$U_2 = \frac{1}{2}\int_V \sigma_r^T [d]\sigma_r dV + \frac{1}{2}\int_V \lambda^T [H]\lambda\, dV + \int_V \sigma_e^T \varepsilon_0 dV +$$
$$+\frac{1}{2}\int_V \sigma_e^T [d]\sigma_e\, dV - \int_{S_u} \mathbf{u}_0^T [A_s]\sigma_e\, dS -$$
$$-\int_V \mathbf{u}_e^T [\mathcal{A}]\sigma_r dV + \int_{S_f} \mathbf{u}_e^T [A_s]\sigma_r dS,$$

where $\sigma_e, \mathbf{u}_e$ are the vectors of stresses and displacements elastic solution of the problem. When the external effect is given, the stress σ_e, displacements $\mathbf{u}_e$ and the third, fourth and fifth members of functional U_2 are fixed, and the last two members of this functional are zero for statically admissible stress. Hence the extremum principle (4) and the problem

$$\frac{1}{2}\int_V \sigma_r^T [d]\sigma_r dV + \frac{1}{2}\int_V \lambda^T [H]\lambda\, dV \Rightarrow \mathbf{min}$$

under the conditions (6)

$$\mathbf{f}_0(C) - \mathbf{f}(\sigma_e + \sigma_r) + [H]\lambda \geq \mathbf{0},\quad \lambda \geq \mathbf{0} \in V;$$
$$-[\mathcal{A}]\sigma_r = \mathbf{0} \in V, \quad [A_s]\sigma_r = \mathbf{0} \in S_f$$

are equivalent to each other. And the problem (5) is equivalent to the extremum principle

$$\frac{1}{2}\int_V \sigma_r^T [d]\sigma_r dV \Rightarrow \mathbf{min}$$

under the conditions (7)

$$\mathbf{f}_0(C) - \mathbf{f}(\sigma_e + \sigma_r) \geq \mathbf{0} \in V,$$
$$-[\mathcal{A}]\sigma_r = \mathbf{0} \in V, \quad [A_s]\sigma_r = \mathbf{0} \in S_f.$$

Generally it corresponds to the principle of a minimum elastic potential of residual forces, formulated by A. Čyras [4,5]:

> *The actual vector of all statically admissible vectors of residual forces in a structure that does not achieve complete plastic failure is the vector for which the elastic potential of these forces is a minimum.*

This principle generalizes the principle of Prager-Symonds [6], which was formulated for a discrete system. The simple load is a separate case of repeated-variable load. That's why the extremum principle (7) can be deduced from analogous extremum principle

$$\frac{1}{2}\int_V \sigma_r^T [d]\sigma_r dV \Rightarrow \mathbf{min}$$

under the conditions (8)

$$-[\mathcal{A}]\sigma_r = \mathbf{0} \in V, \quad [A_s]\sigma_r = \mathbf{0} \in S_f,$$
$$\left.\begin{aligned}\mathbf{f}_0(C) - \mathbf{f}(\sigma_r + \sigma_{ei}^+) \geq \mathbf{0},\\ \mathbf{f}_0(C) - \mathbf{f}(\sigma_r + \sigma_{ei}^-) \geq \mathbf{0}\end{aligned}\right\} \in V,\ i \in I$$

which has been formulated for the body, effected by repeated-variable load [4,5]. Here $\sigma_{ei}^+, \sigma_{ei}^-$ are the extremal stress vectors corresponding to picks i symmetric pair of elastic stress polygons; I is the set elastic stress polygon symmetric picks pairs indices.

It will be shown further that the energy principles of an ideal rigid-plastic body limit equilibrium analysis and optimization can be derived from the extremum principle (5). So not a few extremum principles are connected with the principle of a minimum of total complementary energy as its separate cases. But this principle can be also obtained from the mixed functional by analogy with Castigliano's principle as in the elasticity theory, the latter is obtained from Reissner's functional [15].

Using the Lagrangian multiplier method, the minimization problem (5) can be transformed into the problem of stationary point determination of functional

$$\mathcal{F}_r = \frac{1}{2}\int_V \sigma^T [d]\sigma\, dV + \int_V \mathbf{u}^T [\mathcal{A}]\sigma\, dV + \int_V \sigma^T \varepsilon_0\, dV +$$
$$+\frac{1}{2}\int_V \lambda^T [H]\lambda\, dV + \int_V \beta^T \{\mathbf{f}(\sigma) - \mathbf{f}_0(C) - [H]\lambda\} dV +$$
$$+\int_{S_f} \mathbf{u}^T \{\mathbf{F} - [A_s]\sigma\} dS - \int_{S_u} \mathbf{u}_0^T [A_s]\sigma\, dS, \qquad (9)$$

when $\lambda \geq \mathbf{0},\ \beta \geq \mathbf{0} \in V$. For statically admissible stress this functional has the physical sense of complementary energy. That's why taking the preliminary conditions (2) and (3), the extremum problem (4) and all earlier mentioned energy principles can be derived from functional $\mathcal{F}_r$.

If the second member of functional $\mathcal{F}_r$ is reformed by the Gauss'-Ostrogradski's formula, we will receive the second form of mixed functional and stationary point problem:

$$\mathscr{F}_t = \int_V \sigma^T \left\{ \frac{1}{2}[d]\sigma + \varepsilon_0 - [\mathscr{A}]^T \mathbf{u} \right\} dV + \int_{S_f} \mathbf{u}^T \mathbf{F} dS +$$

$$+ \frac{1}{2}\int_V \lambda^T [H]\lambda \, dV + \int_V \beta^T \{\mathbf{f}(\sigma) - \mathbf{f}_0(C) - [H]\lambda\} dV +$$

$$+ \int_{S_u} \{\mathbf{u} - \mathbf{u}_0\}^T [A_s]\sigma \, dS \Rightarrow stac; \quad \lambda \geq \mathbf{0}, \; \beta \geq \mathbf{0}. \quad (10)$$

Under the conditions of Kuhn-Tucker, the following relationships are for the problems (9) and (10):

$$\begin{aligned} &\mathbf{f}_0(C) - \mathbf{f}(\sigma) + [H]\lambda \geq \mathbf{0} \quad \in V; \\ &-[\mathscr{A}]\sigma = \mathbf{0} \quad \in V, \quad [A_s]\sigma = \mathbf{F} \quad \in S_f; \\ &\beta^T \{\mathbf{f}(\sigma) - \mathbf{f}_0(C) + [H]\lambda\} = \mathbf{0}, \quad \beta = \lambda \quad \in V; \quad (11) \\ &[d]\sigma + \varepsilon_0 + [\nabla \mathbf{f}(\sigma)]^T \lambda - [\mathscr{A}]^T \mathbf{u} = \mathbf{0} \in V; \\ &[A_s]^T \{\mathbf{u} - \mathbf{u}_0\} = \mathbf{0} \quad \in S_u; \quad \lambda \geq \mathbf{0}, \; \beta \geq \mathbf{0} \quad \in V. \end{aligned}$$

Here $[\nabla \mathbf{f}(\sigma)]$ is matrix of yield conditions gradients. This equation system describes the elastic-plastic body stress-strain field. In the relationships (9)-(11), having taken $[H] = [0]$, we will receive ideal elastic-plastic body mixed functionals and the conditions of Kuhn-Tucker, which correspond to the functionals. In case of a linear elastic body ($\lambda = \mathbf{0}, \; \beta = \mathbf{0}$), the functionals $\mathscr{F}_r$ and $\mathscr{F}_t$ become the first and the second form Reissner's functionals.

We are going to show that **the extremum energy principles for the perfectly rigid-plastic body** under monotonically increasing loading [2,3] can be obtained from the principles of a minimum total complementary energy or total deformation energy. Investigating the limit state of the body, we don't estimate elastic deformations and go over to the velocities of displacements and deformations.

The static theorem of simple plastic failure. Let's assume that the velocities of body plastic deformations are fixed as $\dot{\varepsilon}_p = \dot{\varepsilon}_0$, the velocities of initial displacements $\dot{\mathbf{u}} = \mathbf{0}$ and the distribution of plastic constants C is unknown. The yield conditions are also assumed to be uniform. The rate of energy dissipation of deformable body

$$\dot{D} = \int_V \dot{\varepsilon}_p^T \sigma \, dV = \int_V \lambda^T [\nabla \mathbf{f}(\sigma)] \sigma \, dV = \int_V \dot{\Theta} C \, dV,$$

where $\dot{\Theta}$ is the fixed field of plastic deformations velocities intensity. Then the following problem is obtained from the extremum principle (5):

$$\int_V \dot{\Theta} C \, dV \Rightarrow \mathbf{min}$$

under the conditions (12)

$$\begin{aligned} &\mathbf{f}_0(C) - \mathbf{f}(\sigma) \geq \mathbf{0}, \quad C \geq 0 \quad \in V; \\ &-[\mathscr{A}]\sigma = \mathbf{0} \quad \in V, \quad [A_s]\sigma = \mathbf{F} \quad \in S_f. \end{aligned}$$

It corresponds to the following extremum principle:

The velocities intensity of plastic deformations being fixed, the actual field of all statically admissible stress fields in a simple plastic failure is the one for which the energy dissipation rate is a minimum.

So the problem (12) in the case of fixed plastic deformations velocities intensity expresses the static theorem of simple plastic failure [2] formulated by A. Čyras. It allows to determine the optimum distribution of plastic constant or the parameter of plastic constant C_0 (under the law $C = \rho C_0$).

The static theorem of the limit load. Let's say, the initial deformations do not exist ($\varepsilon_0 = \mathbf{0}$), the distribution of external load is unknown (the direction is known) and velocities of displacements at the place of load operation are fixed. The surface of the body , where the displacements or their velocities are determined and unknown forces act, is signed as S_u. Hence the body surface $S \equiv S_u = S_{u1} \cup S_{u2}$. At the surface S_{u1} the power of operating external forces

$$\dot{W} = \int_{S_{u1}} \dot{\mathbf{u}}_0^T \mathbf{F} \, dS = \int_{S_{u1}} \dot{\mathbf{u}}_0^T [A_s] \sigma \, dS.$$

Paying attention to these conditions the problem (5) takes the form:

$$\int_{S_{u1}} \dot{\mathbf{u}}_0^T \mathbf{F} \, dS \Rightarrow \mathbf{max}$$

under the conditions (13)

$$\begin{aligned} &\mathbf{f}(\sigma) \leq \mathbf{f}_0(C), \quad -[\mathscr{A}]\sigma = \mathbf{0} \quad \in V; \\ &-[A_s]\sigma + \mathbf{F} = \mathbf{0}, \quad -\mathbf{F} \leq \mathbf{0} \quad \in S_{u1}. \end{aligned}$$

This problem in the case of fixed displacements velocities expresses the static theorem of the limit load formulated by A. Čyras [2]:

The actual functions of all statically admissible stress functions in a simple plastic failure are those which maximizes the power of external loading.

The problem allows to determine the distribution of optimum limit load or the parameter F_0 of limit load, when the law of load distribution is $\mathbf{F} = \eta F_0$. In this case the problem (13) expresses Gvozdev's theorem [16].

3. The principles of a minimum of total strain energy

The total strain energy of linearly strengthening elastic-plastic body:

$$\Pi = \frac{1}{2}\int_V \sigma^T[d]\sigma\, dV - \int_{S_f} \mathbf{u}^T \mathbf{F}\, dS + \int_V \lambda^T[\nabla \mathbf{f}(\sigma)]\sigma\, dV - \frac{1}{2}\int_V \lambda^T[H]\lambda\, dV. \quad (14)$$

The first two members of this functional express the total potential energy of the elastic body, while the third member expresses the energy dissipation. The last two equations of (11) determine the kinematically admissible vectors of the displacements $\mathbf{u}$. In the case of kinematically admissible displacements, the functional $\mathscr{F}_t$, multiplied by -1, expresses the total deformation energy of elasto-plastic body. That's why the problem of stationary point determination of functional (10) can be transformed into the maximization problem

$$-\frac{1}{2}\int_V \sigma^T[d]\sigma\, dV - \int_V \lambda^T[\nabla \mathbf{f}(\sigma)]\sigma\, dV + \frac{1}{2}\int_V \lambda^T[H]\lambda\, dV + \int_{S_f} \mathbf{u}^T \mathbf{F}\, dS + \int_V \lambda^T\{\mathbf{f}(\sigma) - \mathbf{f}_0(C) - [H]\lambda\} dV \Rightarrow \max$$

under the conditions (15)

$$[d]\sigma + \varepsilon_0 + [\nabla \mathbf{f}(\sigma)]^T \lambda - [\mathscr{A}]^T \mathbf{u} = \mathbf{0} \quad \in V,$$
$$[A_s]^T\{\mathbf{u} - \mathbf{u}_0\} = \mathbf{0} \quad \in S_u, \quad \lambda \geq \mathbf{0} \quad \in V$$

or the minimization problem

$$\Pi - \int_V \lambda^T\{\mathbf{f}(\sigma) - \mathbf{f}_0(C) + [H]\lambda\} dV \Rightarrow \min$$

under the conditions (16)

$$[d]\sigma + \varepsilon_0 + [\nabla \mathbf{f}(\sigma)]^T \lambda - [\mathscr{A}]^T \mathbf{u} = \mathbf{0} \quad \in V,$$
$$[A_s]^T\{\mathbf{u} - \mathbf{u}_0\} = \mathbf{0} \quad \in S_u, \quad \lambda \geq \mathbf{0} \quad \in V.$$

These problems correspond to the extremum principle:

The actual vector of all kinematically admissible vectors of displacements in a body is the one for which the total potential energy of deformations is a minimum.

This principle for the perfectly elastic-plastic body is expressed by the following extremum problem:

$$\int_{S_f} \mathbf{u}^T \mathbf{F}\, dS - \frac{1}{2}\int_V \sigma^T[d]\sigma\, dV - \int_V \lambda^T[\nabla \mathbf{f}(\sigma)]\sigma\, dV + \int_V \lambda^T\{\mathbf{f}(\sigma) - \mathbf{f}_0(C)\} dV \Rightarrow \max$$

under the conditions (17)

$$[d]\sigma + \varepsilon_0 + [\nabla \mathbf{f}(\sigma)]^T \lambda - [\mathscr{A}]^T \mathbf{u} = \mathbf{0} \quad \in V,$$
$$[A_s]^T\{\mathbf{u} - \mathbf{u}_0\} = \mathbf{0} \quad \in S_u, \quad \lambda \geq \mathbf{0} \quad \in V.$$

The problems (4),(15) and (5),(17) form the dual pairs of stress-strain field analysis problems in static and kinematic formulations for the strengthening and perfectly elastic-plastic body. When plastic deformations are known (fixed), the problem (17) expresses the Greenberg's principle [9,17]. In case of linear elastic structure this problem expresses the classical variational principle of total potential energy (Lagrangian principle).

Using the Lagrangian multiplier method, the dual kinematic formulation for the static problem (6) can be created:

$$-\frac{1}{2}\int_V \sigma_r^T[d]\sigma_r dV - \frac{1}{2}\int_V \lambda^T[H]\lambda\, dV - \int_V \lambda^T[\nabla \mathbf{f}(\sigma_e + \sigma_r)]\sigma_r dV + \int_V \lambda^T\{\mathbf{f}(\sigma_e + \sigma_r) - \mathbf{f}_0(C) - [H]\lambda\} dV \Rightarrow \max$$

under the conditions (18)

$$[d]\sigma_r + [\nabla \mathbf{f}(\sigma_e + \sigma_r)]^T \lambda - [\mathscr{A}]^T \mathbf{u}_r = \mathbf{0}, \ \lambda \geq \mathbf{0} \quad \in V;$$
$$[A_s]^T \mathbf{u}_r = \mathbf{0} \quad \in S_u.$$

It expresses the principle of a minimum of total residual deformation energy:

Of all kinematically admissible vectors of residual displacements in a system that does not achieve the *complete plastic failure the actual vector is one for which the energy of residual deformations is a minimum.*

Taking $[H]=[0]$, the extremum problem (18) expresses the principle of a minimum of residual deformation energy for perfectly elastic-plastic body.

Now we are going to investigate the kinematic theorems for perfectly rigid-plastic body. They also can be obtained from the principles of a minimum of total complementary energy or total strain energy.

The kinematic theorem of simple plastic failure. We create the dual kinematic formulation for the problem (12) using Lagrangian multiplier method:

$$\int_{S_f} \dot{\mathbf{u}}^T \mathbf{F}\, dS - \int_V \dot{\lambda}^T \left\{ [\nabla \mathbf{f}(\sigma)]\sigma - [\nabla \mathbf{f}_0(C)]C \right\} dV + \\ + \int_V \dot{\lambda}^T \{\mathbf{f}(\sigma) - \mathbf{f}_0(C)\} dV \Rightarrow \mathbf{max}$$

under the conditions (19)

$$[\nabla \mathbf{f}_0(C)]^T \dot{\lambda} \le \dot{\Theta}, \quad \dot{\lambda} \ge \mathbf{0} \quad \in V;$$
$$[\nabla \mathbf{f}(\sigma)]^T \dot{\lambda} - [\mathscr{A}]^T \dot{\mathbf{u}} = \mathbf{0} \quad \in V,$$
$$[A_s]^T \dot{\mathbf{u}} = \mathbf{0} \quad \in S_u.$$

The first member of the objective function of this problem means the power of external load, the second and the third members in case of optimum solution are equal to null and the constraints of the objective function determine the sphere of kinematically admissible displacements velocities.

Hence the problem (19) expresses the kinematic theorem of simple plastic failure formulated by A. Čyras [2]:

The actual vector of all kinematically admissible displacements velocity vectors in a simple plastic failure is the one for which the power of the external loading is a maximum.

The first condition (inequality) of the problem (19) has the physical meaning of the intensity of plastic deformation velocities and at once the physical meaning of energy dissipation rate constraint. Hence we can get this formulation of extremum principle directly from the problem (17) having fixed the energy dissipation rate. When the law of plastic constant distribution is given $C = \rho C_0$ and the extremum problems (12) and (19) expresses the static and kinematic theorems of the plastic constant multiplier [1, 2].

The kinematic theorem of the limit load. The static formulation of the problem (19) can be rearranged into the kinematic formulation by the Lagrangian multiplier method:

$$\int_V \dot{\lambda}^T [\nabla \mathbf{f}(\sigma)] \sigma\, dV - \int_V \dot{\lambda}^T \{\mathbf{f}(\sigma) - \mathbf{f}_0(C)\} dV \Rightarrow \mathbf{min}$$

under the conditions (20)

$$-[\nabla \mathbf{f}(\sigma)]^T \dot{\lambda} + [\mathscr{A}]^T \dot{\mathbf{u}} = \mathbf{0}, \quad \dot{\lambda} \ge \mathbf{0} \quad \in V;$$
$$\dot{\mathbf{u}} \ge \dot{\mathbf{u}}_0 \quad \in S_{u1}; \qquad [A_s]^T \dot{\mathbf{u}} = \mathbf{0} \quad \in S_{u2}.$$

The first member of an objective expresses the rate of energy dissipation and the second one for the optimal solution of problem is zero. Hence the objective function has the physical meaning of the rate of energy dissipation. So the problem (20) corresponds to the kinematic theorem of a simple plastic failure formulated by A. Čyras [2]:

The actual vector of all kinematically admissible vectors of displacement velocities in a simple plastic failure is the one for which the rate of energy dissipation is a minimum.

The fixing of displacement velocities means a constraint for the load power. Therefore the kinematic theorem of the limit load can be deduced directly from the problem (17) fixing the load power. When $\mathbf{F} = \eta F_0$ this extremum principle become the kinematic theorem of the limit load multiplier [1, 2].

In Fig.1 the all relations of extremum energy principles for deformable body are presented.

4. Conclusions

The extremum energy principles of the elastic-plastic body stress-strain analysis and limit equilibrium problems in the case of monotonically increasing (simple) loading have been investigated. The principle of a minimum of total complementary energy (4) and the principle of a minimum of total deformation energy (16) have been formulated. It has been shown that the extremum principles of Haar-Karman, Colonnetti, Prager-Symonds, Castigliano

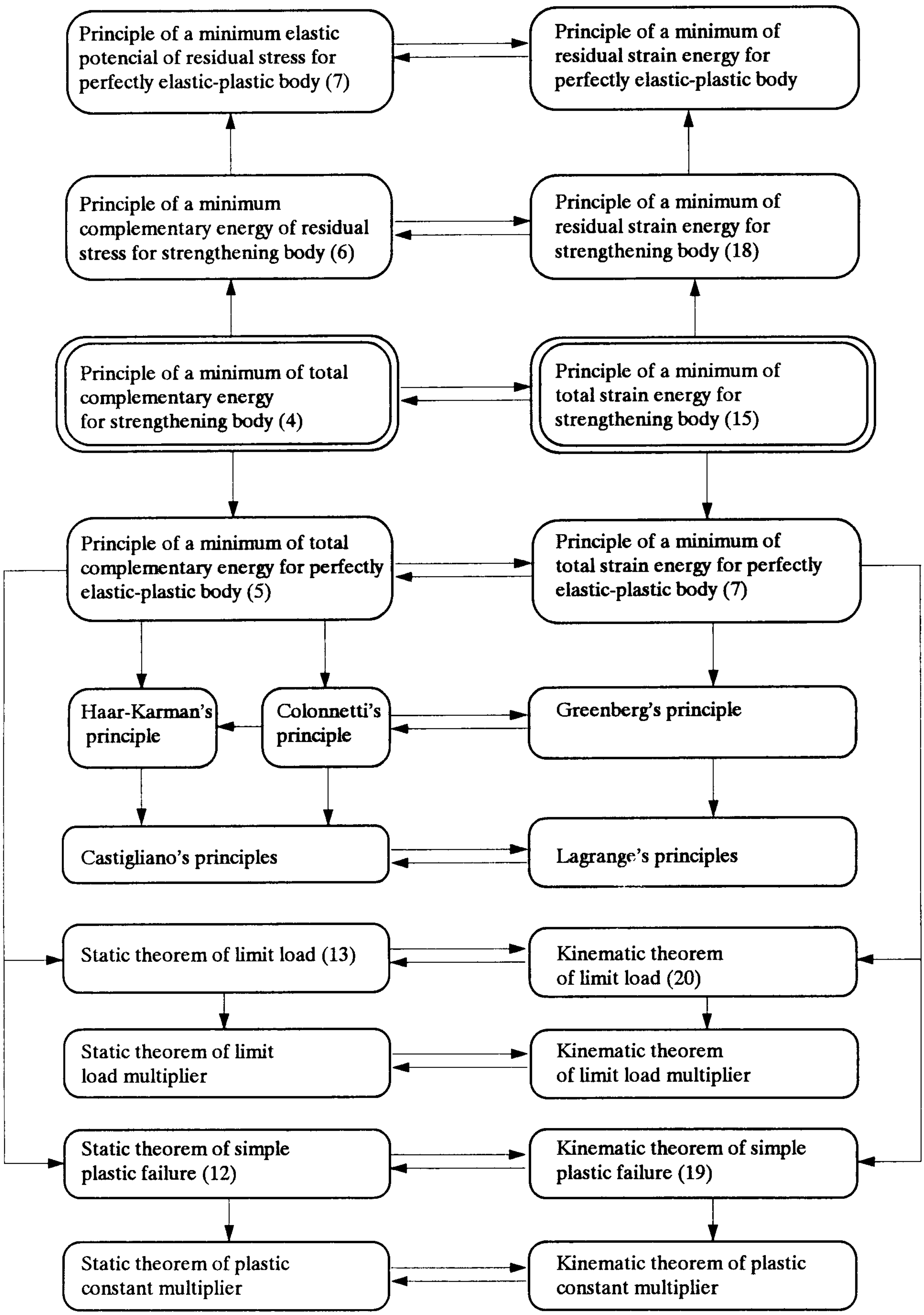

Fig. 1. Relations of extremum energy principles of deformable body; ⇄ dual relations

and Menabrea are received from the principle of a minimum of total complementary energy and the principles of Greenberg and Lagrange are received from the principle of a minimum of total deformation energy. The energy principles of ideal rigid-plastic body limit state analysis are derived from the principle of a minimum of total complementary energy. It has been also shown, that the kinematic theorems of the limit load and the simple plastic failure [2] can be received from the elastic-plastic body principle of a minimum of total deformation energy.

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DEFORMUOJAMO KŪNO EKSTREMINIŲ ENERGINIŲ PRINCIPŲ RYŠIAI IR TRANSFORMACIJOS

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Santrauka

Išnagrinėti tampraus plastiško kūno įtempimų ir deformacijų analizės ir ribinės pusiausvyros būvio ekstreminiai energiniai principai paprasto (vienkartinio) apkrovimo atveju. Suformuluotas pilnutinės papildomosios energijos minimumo principas (4) ir pilnutinės deformacijų energijos minimumo principas (14), atsižvelgiant į pradines deformacijas ir medžiagos stiprėjimą. Parodyta, kad Haro-Karmano, Kolonečio, Pragerio-Saimondso, Kastiljano bei Menabrea ekstreminiai principai gaunami iš pilnutinės papildomosios energijos minimumo principo, o Grinbergo ir Lagranžo principai - iš pilnutinės deformavimo energijos minimumo principo. Idealiai standaus plastiško kūno ribinio būvio analizės energiniai principai išvesti iš pilnutinės papildomosios energijos minimumo principo. Taip pat parodyta, kad kinematinės teoremos apie ribinę apkrovą ir apie paprastą plastinį suirimą gali būti gautos iš tampraus plastiško kūno pilnutinės deformavimo energijos minimumo principo.

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