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A POSTPROCESSED ERROR ESTIMATION IN THE FINITE ELEMENT ANALYSIS

R. Baušys

1. Introduction

An error estimation has become a subject of an intensive research in the field of the finite element analysis. The error estimation techniques assess the amount of solution error due to finite element mesh discretization. The assessment of the discretization errors can be performed in two ways: a priori and posteriori error estimation. A priori error estimates provide only qualitative assessment of the solution and finite element mesh such as the smoothness of the solution, the regularity of the mesh, asymptotic rate of convergence which is essential for theoretical investigation. A posteriori error estimates can be utilised to give more specific assessment of errors in different measures. A posteriori error estimation can be carried out in various ways, and well-known approaches for control of the discretization error can be classified into main categories as follows:

1) Element residual methods. The residual is the function defining the measure of how much the approximate solution fails to satisfy governing equations and boundary conditions. The residual is computed over each element and used as the initial information in local problems in order to determine the local error [1-3]. They strongly depend upon the governing operators and thus require care in extensions to more complex non-linear problems.

2) Subdomain-residual methods. Here the local problem for the error of residual in an element of the interest is formulated over a patch of the elements surrounding the element, which is computed in terms of the solution of local problems of a higher-order finite element approximation of the original problem [4].

3) Duality methods. These methods, valid for self-adjoint ellliptic problems, use duality theory of convex optimization. There a primal and dual problem for the element error are obtained which provide upper and lower bounds of the local element errors [5].

4) Interpolation methods. These error estimates were derived using interpolation theory and the best approximation property of the finite element method. They generally require a posteriori error estimation of higher-order derivatives and hence rely heavily upon superconvergence properties. More specifically, the estimates utilise that the finite element error is bounded by the interpolation error of which the error estimate is well established [6-7].

5) Extrapolation methods. Here the estimation of the error is done comparing two global solutions which are computed on the same mesh but differ by one polynomial order. This approach is based on Richardson's extrapolation [7].

6) Postprocessed error estimators. The essence of these error estimators is to replace the exact solution with a postprocessed solution. The improved postprocessed flux is constructed from the original finite element solution by some postprocessing techniques, alternatively called recovery procedures. Therefore, the reliability of the error estimators of this type primarily depends on the quality of the recovered solution [8-10].

Recently, an objective methodology for assessing the reliability of a posteriori error estimators has been developed by Babuška et al. [11].

In this paper we focus our attention on the application of the Superconvergent Patch Recovery technique for displacements (SPRD) to the elliptic problems which appear in elastostatics. The SPRD technique is based on a higher order displacement field fitted to a superconvergent values in a least squares sense over local element patches. This approach was implemented for the error estimation in free vibration problems [12-13].

The numerical experiments reported in this paper cover examples of two-dimensional linear quadrilateral and triangular elements. Numerical results show for the postprocessed solution an improved accuracy and improved convergence rate of the error of the improved solution compared to the FE solution and the other postprocessed error estimation techniques.

2. A postprocessed error estimation

The problems discussed in this paper can be described in the form of

$$Lu + f = 0 \qquad in\,\Omega \tag{1}$$

together with appropriate boundary conditions. Here L is a differential operator acting on the unknown function u , f denotes some function which is known inside the domain Ω.

In order to investigate the accuracy of the solution, we need to discuss the dicretization error. The point-wise discretization error is simply the difference between exact solution and the finite element solution:

$$\boldsymbol{e}_u = \boldsymbol{u} - \boldsymbol{u}^h \tag{2}$$

where $\boldsymbol{u}$ is the exact solution and $\boldsymbol{u}^h$ is the corresponding FE-approximation.

The point-wise discretization error is difficult to interpret, so certain norms to measure error are used to assess finite element approximation. One of the most popular measurements of the discretization error is based on the energy norm, expressed as

$$\|e\| = \left(\int_\Omega e_u^T L e_u \, d\Omega\right)^{\frac{1}{2}} \tag{3}$$

Traditional error estimates for finite element methods are a priori bounds, predicting the asymptotic rate of convergence as the size of elements tends to zero which can be expressed

$$\|e\| \leq C h^p \|u\|_{p+1} \tag{4}$$

where C is a constant, h is the characteristic size of the finite elements, p is the polynomial order of the finite element space and $\|u\|_{p+1}$ can be expressed as

$$\|u\|_{p+1}^2 = \int_\Omega \left[\left(u^{(p+1)}\right)^2 + \left(u^{(p)}\right)^2 + \ldots + (u)^2\right] d\Omega \tag{5}$$

which reflects the smoothness of the exact solution. Presence of any singularity in the problem affects the degree to which the solution is smooth and therefore the rate of convergence.

The essence of the postprocessed error estimator is to replace the exact solution with a postprocessed solution of higher quality:

$$\boldsymbol{e}_u \approx \bar{\boldsymbol{e}}_u = \boldsymbol{u}^* - \boldsymbol{u}^h \tag{6}$$

where $\bar{e}_u$ is the point-wise estimated error. Using the improved solution we have an estimation of Eq.(3)

$$\|\bar{e}\| = \left(\int_\Omega \bar{e}_u^T L \bar{e}_u \, d\Omega\right)^{\frac{1}{2}} \tag{7}$$

In practice, we calculate this norm by summing over all elements in the domain Ω:

$$\|\bar{e}\|^2 = \sum_{i=1}^{nel} \|\bar{e}\|_i^2 = \sum_{i=1}^{nel} \int_{\Omega_i} \bar{e}_u^T L \bar{e}_u \, d\Omega_i \tag{8}$$

where Ω_i is an element domain and nel is the total number of elements.

The quality of any error estimator is dictated by an effectivity index:

$$\theta = \frac{\|\bar{e}\|}{\|e\|} = \frac{\|u^* - u^h\|}{\|u - u^h\|} \tag{9}$$

It has been known that if the postprocessed solution exhibits superconvergence property, meaning that the rate of convergence of the postprocessed solution is at least one order higher than finite element solution, it can be shown that the postprocessed type of the error estimator is asymptotically exact [9]. By asymptotical exactness, we understand that as the characteristic size of the elements $h \to 0$, the effectivity index converges to unity i.e. $\theta \to 1$. Alternatively, this condition can be expressed as

$$\left(1 - \frac{\|u - u^*\|}{\|u - u^h\|}\right) \leq \theta \leq \left(1 + \frac{\|u - u^*\|}{\|u - u^h\|}\right) \tag{10}$$

Assuming that the true error converges as $\|u - u^h\| = C_h h^p$ and the error of the postprocessed solution $\|u - u^*\| = C_* h^{p+\alpha}$ for some superconvergent solution with $\alpha \geq 1$, we can obtain

$$1 - C h^\alpha \leq \theta \leq 1 + C h^\alpha \tag{11}$$

Showing that the effectivity index approaches unity as $h \to 0$. In the above, $\alpha \geq 1$ indicates whether the recovered solution has a higher rate of convergence than the finite element solution.

3. An improved solution by patch recovery

The construction of an improved solution based on superconvergence phenomenon is explained in the following part. The main idea is that the approximate solution, the primary function (displacements) or its derivatives (stresses) exhibits higher rate of convergence at some specific points than the global rate. These points are called the superconvergent points of the finite element solution. It has been known that the nodal points of the finite element approximation are found to be the exceptional points at which the prime variables (displacements) have higher order accuracy in respect of the global accuracy [14]. Thus, applying a recovery technique which consists of a least squares fit of a local polynomial to displacement values at higher accuracy points, we obtain displacement field of the superior accuracy. The recovered displacement field over an element $\tau \in T_h$ is constructed as

$$u^*(x) = \sum_r N_r^*(x)(u_r^*) + \sum_s N_s^*(x)(u_s^*) \quad (12)$$

where r is used to denote finite element τ nodes, s denotes additional nodes of the element of the recovered displacement field, $N_r^*(x)$ and $N_s^*(x)$ are local basis functions of the order $p+1$ associated with the original element nodes and the additional ones, respectively.

The nodal values of the original finite element displacements are of the superior accuracy and are assumed fixed $(u_r^*) \equiv (u_r^h)$. The recovered displacement values (u_s^*) at the additional nodes are obtained by solving least squares problem in the reduced element patch Ω_τ which represents the union of the element under consideration and the part of the surrounding elements:

Find $u^* \in P_{p+1}$ such that

$$J_{\Omega_\tau}(u^*) = \min_{(u^{f^*}) \in P_{p+1}} J_{\Omega_\tau}(u^{f^*}) \quad (13)$$

where

$$J_{\Omega_\tau}(u^{f^*}) = \sum_{j=1}^{ns} w_j^2 R_u^T(x_j) R_u(x_j) \quad (14)$$

where the residual $R_u(x_j)$ is defined by expression as

$$R_u = (u^{f^*}) - (u_r^h) \quad (15)$$

and

$$(u^{f^*}) = [Q(x)]b \quad (16)$$

Here x_j is the location of j-th sampling point in the element patch Ω_τ, w_j is the weight assigned to the j-th sampling point and ns is the total number of the sampling (nodal) points in the element patch Ω_τ and b are unknown coefficients. $[Q(x)]$ contains the appropriate polynomial terms of $p+1$ order. For the 2D-case and for instance linear triangle, $Q(x)$ is given by:

$$Q = [1, x, y, x^2, xy, y^2] \quad (17)$$

and

$$b = [b_1, b_2, b_3, b_4, b_5, b_6]^T \quad (18)$$

The unknown parameters b are determined by the solution of the weighted least squares problems which gives us the system of the linear equations

$$\left(\sum_j w_j^2 [Q(x)]^T [Q(x)]\right) b = \sum_j w_j^2 [Q(x)]^T u_j^h \quad (19)$$

where w_j is a weighting function. The weighting function is a positive continuous function which is the unity for the element defining the patch and decreases monotonically with increasing distance away from master element. In this study, we used

$$w_j = \frac{(d_m - d_j - c)}{d_m} \quad (20)$$

where d_m is the maximum distance from the centre of the master element of the patch to sampling points of the patch, d_j is the distance from the centre of the master element to j-th sampling point and c is the positive constant; we have used c=0.01.

In order to maintain locality of the least squares fit we use a reduced element patch with the size of $2h$ in the present patch recovery technique for displacements. Defining the construction of reduced element patch, for each element $\tau \in T_h$ we denote be $E(\tau)$ the set of its edges. So for each element τ the patch Ω_τ which consists of the part of elements surrounding the master element is denoted by

$$\Omega_\tau = \bigcup_{\tau \in E^*(\tau')} \tau' \quad (21)$$

For the triangular elements, set $E^*(\tau')$ coincides with $E(\tau)$, and for quadrilateral elements, set $E^*(\tau')$ consists of the adjacent edges connected to one of the nodes of the element ($E^*(\tau') \subset E(\tau)$). The details concerning construction of the reduced element patch can be found in [12-13].

We observe that the recovered displacement field which can be obtained by solving least squares problem (13) can be discontinuous over the element boundaries. In order to determine a continuous recovered displacement field we propose a simple averaging for the points at the element boundaries of the overlapping patch solutions.

When the recovered displacements are determined over all elements $\tau \in T_h$, we obtain the displacement field of the higher accuracy and a postprocessed error estimation can be performed.

4. Numerical example

We consider the model problem governed by the differential equation

$$-\Delta u = f \qquad in\,\Omega \tag{22}$$

where $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the Laplace operator, u is unknown function , f is some given data, and the Dirichlet boundary condition with prescribed values

$$u = u_b \qquad on\,\partial\Omega \tag{23}$$

where $\partial\Omega$ denotes boundary of the unit square domain $\Omega = (0,1) \times (0,1)$. We prescribe the boundary data and choose the 'load' function f that corresponds to an exact solution of the form

$$u(x,y) = x(1-x)y(1-y)(1+2x+7y) \tag{24}$$

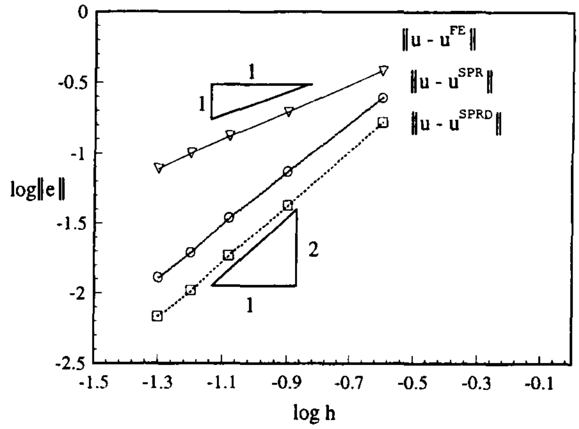
This problem has been used by Zienkiewicz and Zhu [8] to demonstrate the properties of the proposed error estimator. The derivatives are defined:

$$\sigma = \nabla u \quad with\ \sigma_x = \frac{\partial u}{\partial x} \quad and\ \sigma_y = \frac{\partial u}{\partial y} \tag{25}$$

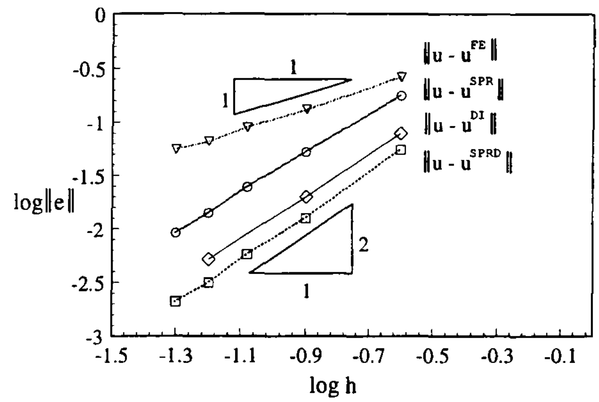
The regular meshes are used in the computational experiments. Numerical results are presented for both quadrilateral and triangular linear elements in order to test whether the proposed SPRD procedure is adequate to estimate the error of the finite element solution.

The following labels are used in figures and text: FE for the original finite element solution, SPR for the Zienkiewicz and Zhu [8] superconvergent patch recovery technique, SPRD for the superconvergent patch recovery technique for displacements presented here and DI for the displacement interpolation recovery technique developed in [15].

The error in energy of recovered solutions obtained using different recovery procedures is compared with that of finite element solution in Fig 1. For the sake of comparison the results obtained by DI technique are also presented in the case of quadrilateral elements in Fig 1b. The improved derivatives obtained by SPRD technique exhibits $O(h^{p+1})$ superior rate of convergence as expected and shows higher accuracy than SPR and DI approaches. So the proposed SPRD technique provides asymptotically exact error estimate.



a) triangular element



b) quadrilateral element

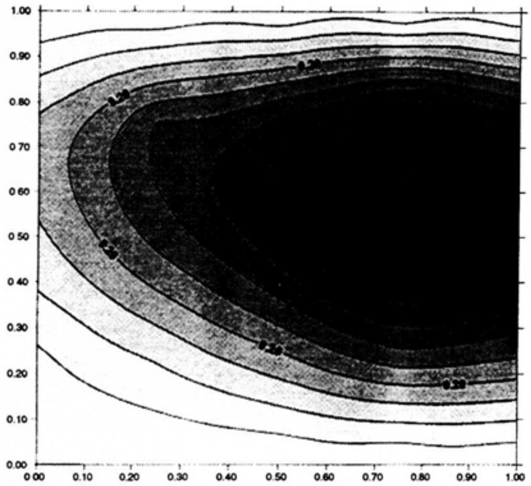
Fig 1. Convergence of the energy norm of error

In Figs 2 and 4, the point-wise error distribution of the derivative component σ_x is shown obtained using the finite element method and recovery approaches SPR and SPRD using linear quadrilateral and triangular elements.

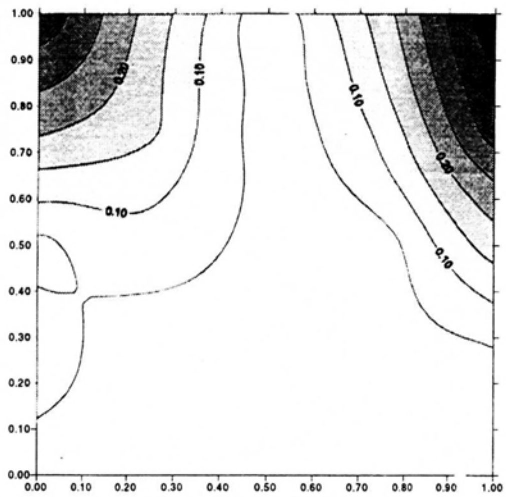
The point-wise error distribution of the derivative component σ_y is presented in Figs 3 and 5 obtained using the same solution procedures.

5. Conclusions

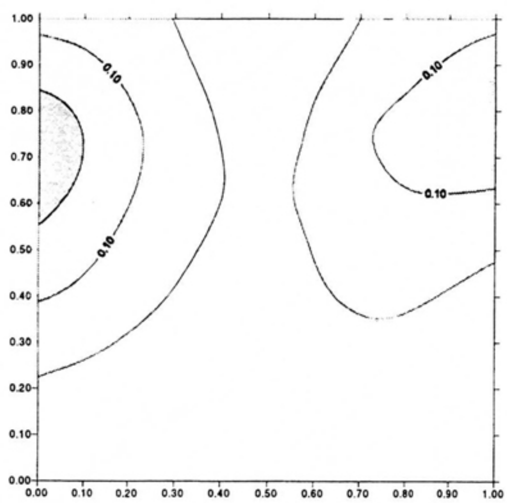
A method for obtaining postprocessed solution of the higher order accuracy has been presented. The proposed SPRD technique is essentially a least square fit of the prime variables (displacements) at superconvergent points. This approach provides suiperconvergent displacement field u_i^* over local reduced element patches. Since the SPRD technique recovers superconvergent displacement field that is at least one order higher than finite element solution, the method can successfully be implemented in error estimation of the finite element solution. The proposed approach



a) σ_x -FE

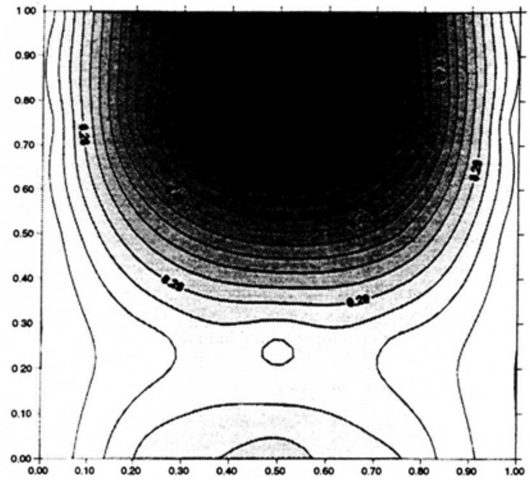


b) σ_x -SPR

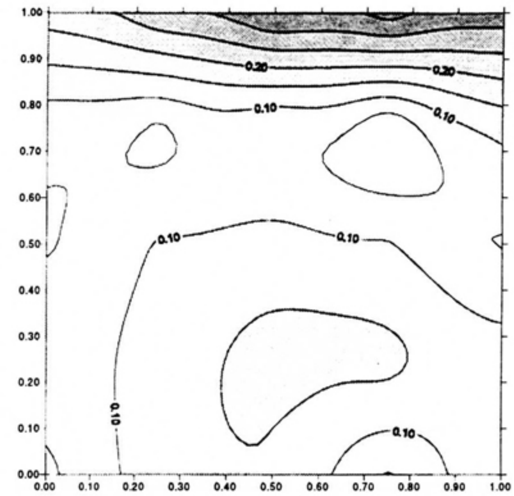


c) σ_x -SPRD

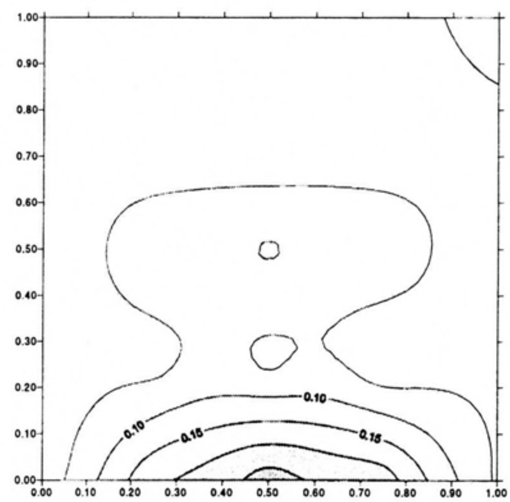
Fig 2. Distribution of absolute error in flux component σ_x using quadrilateral elements



a) σ_y -FE

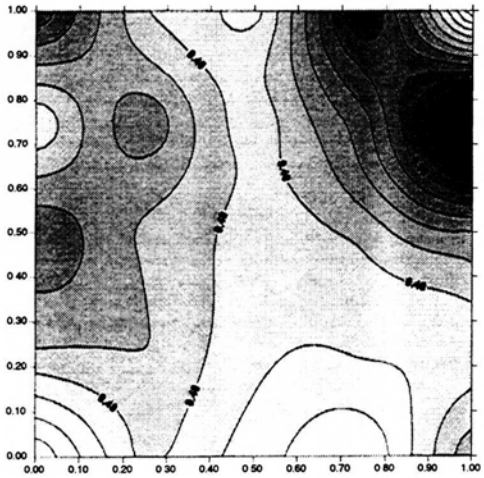


b) σ_y -SPR

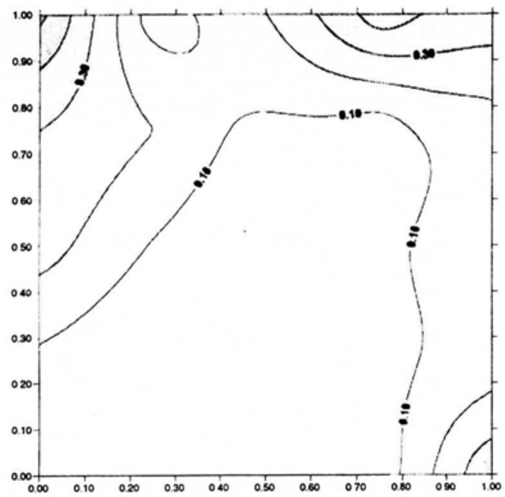


c) σ_y -SPRD

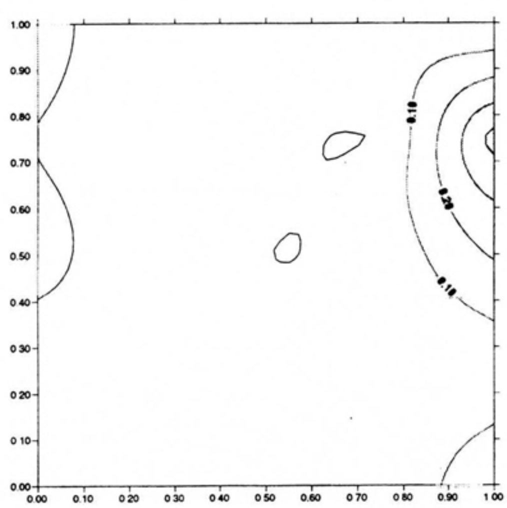
Fig 3. Distribution of absolute error in flux component σ_y using quadrilateral elements



a) σ_x -FE

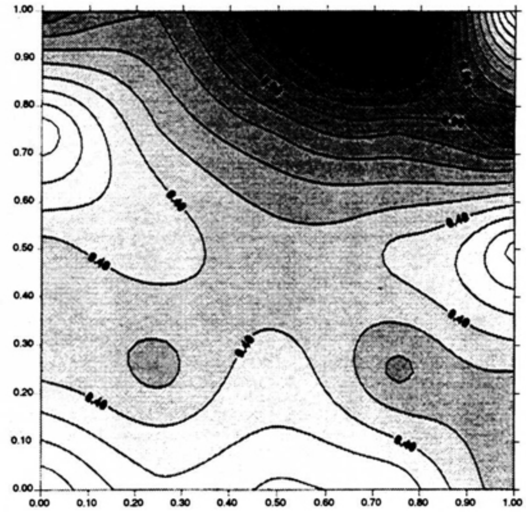


b) σ_x -SPR

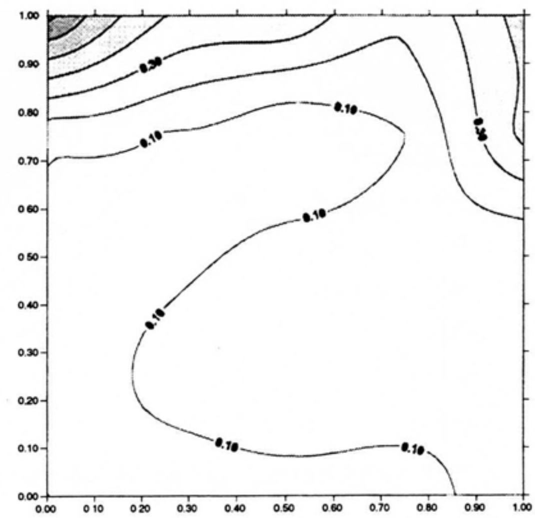


c) σ_x -SPRD

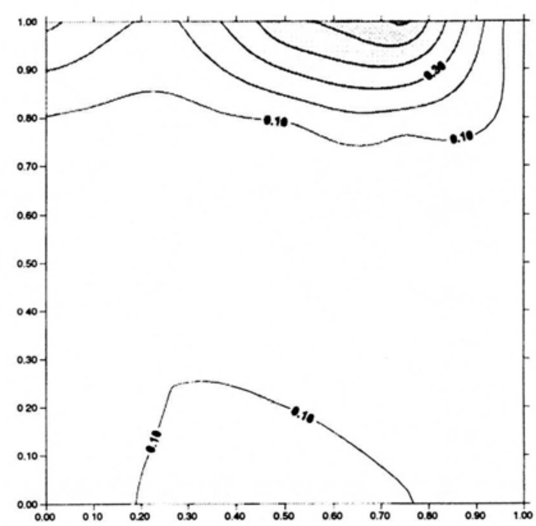
Fig 4. Distribution of absolute error in flux component σ_x using triangular elements



a) σ_y -FE



a) σ_y -SPR



c) σ_y -SPRD

Fig 5. Distribution of absolute error in flux component σ_y using triangular elements was devised in such a way that it works locally but not globally. This entails that the cost involved using this approach is small compared to that of the finite element computations.

As evidenced by numerical experiments, a powerful technique has been developed for recovery of improved solutions. The obtained numerical results are compared with the results of the most popular error estimator developed by Zienkiewicz and Zhu.

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POPROCESORINIS PAKLAIDŲ NUSTATYMAS BAIGTINIŲ ELEMENTŲ ANALIZĖJE

R. Baušys

Santrauka

Straipsnyje pateikiamas originalus metodas poprocesoriniam aukštesnės tikslumo klasės sprendiniui gauti. Šis būdas paremtas superkonvergavimo pradinio baigtinių elementų sprendinio savybėmis. Įrodyta, kad kiekviename elemente egzistuoja taškai, kuriuose baigtinių elementų sprendinys turi didesnį konvergavimo greitį palyginti su globaliu visai diskrečiajai erdvei įrodomu konvergavimo greičiu. Šie taškai vadinami superkonvergavimo baigtinių elementų sprendinio taškais. Poslinkių lauko superkonvergavimo taškai sutampa su baigtinių elementų mazgo taškais. "Superkonvergavimo lopinio atstatymo" būdas originaliai buvo panaudotas aukštesnės tikslumo klasės gradientų laukų (įtempčių, deformacijų) radimui.

Šiame darbe pateikta šio "superkonvergavimo lopinio atstatymo" būdo versija, pritaikyta tiesioginiams aproksimacijos kintamiesiams (poslinkiams). Pagrindinę pasiūlyto metodo idėją sudaro atstatyto poslinkių lauko interpoliavimas naudojant superkonvergavimo poslinkių reikšmes mažiausių kvadratų būdu. Su gautu poprocesoriniu sprendiniu gali būti įvertinta tikroji baigtinių elementų sprendinio paklaida bei jos pasiskirstymas tyrinėjamoje konstrukcijoje.

Skaitiniai eksperimentai atlikti naudojant tiesinius keturkampius ir trikampius baigtinius elementus. Atlikus skaitinių rezultatų analizę, galima padaryti šias išvadas:

1. Poprocesorinio sprendinio paklaidos, išreikštos energine norma, konvergavimo greitis yra viena eile aukštesnis nei pradinio baigtinių elementų sprendinio.

2. Poprocesorinio sprendinio, gauto pasiūlytu metodu, paklaidos yra mažesnės už paklaidas, gautas viena iš populiariausių procedūrų, sukurta Zienkiewicziaus ir Zhu.

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